MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ____

RIN: ____

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Section: ____

Answer **ALL** questions. Open Orange DMC-textbook. Nothing else.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	Total
150	25	25	25	25	250

1 Circle one answer per question. 10 points for each correct answer.

(1) Give the negation of "Attending class is necessary to pass the course".

- [A] IF you did not pass the course THEN you did not attend class.
- [B] IF you did not attend class THEN you did not pass the course.
- [C] You did attend class AND you did not pass the course.
- [D] You did not attend class AND you did pass the course.
- [E] None of the above.

D

$p \equiv$ attend $q \equiv$ pass

$\neg p \to \neg q$ (also $q \to p$)

$\neg(\neg p \to \neg q) \equiv \neg(p \vee \neg q)$

$\equiv \neg p \wedge q$

$\equiv$ dont attend AND pass.

(2) Compute $S = \sum_{i=1}^{4}\sum_{j=1}^{4}(ij)$.

- [A] $S = 50$.
- [B] $S = 100$.
- [C] $S = 200$.
- [D] $S = 400$.
- [E] None of the above.

B.

Inner Sum first (Constant Rule)

$\sum_{j=1}^{4} ij \equiv i \sum_{j=1}^{4} j \equiv i \times \frac{4\times5}{2} = 10i$

$\sum_{i=1}^{4} 10i = 10 \underbrace{\sum_{i=1}^{4} i}_{10} = 10 \times 10 = \boxed{100}$

(3) Estimate this sum $S = \sum_{i=0}^{5}\sum_{j=0}^{10} 2^{i+j}$.

- [A] $S = 3.2 \times 10^3$.
- [B] $S = 1.3 \times 10^4$.
- [C] $S = 2.6 \times 10^4$.
- [D] $S = 1.3 \times 10^5$.
- [E] $S = 2.6 \times 10^5$.

D

$\sum_{j=0}^{10} 2^{i+j} = \sum_{j=0}^{10} 2^i 2^j = 2^i \sum_{j=0}^{10} 2^j = 2^i(2^{11}-1)$

$\sum_{i=0}^{5} (2^{11}-1)2^i = (2^{11}-1)\sum_{i=0}^{5} 2^i = (2^{11}-1)(2^6-1)$

$\approx 2^{11} \cdot 2^6$

$= 2^{17} \approx 2^7 \cdot 2^{10} \approx 2^7 \times 1000$

$= 128 \times 1000 \approx \boxed{1.3 \times 10^5}$

(4) Let $S(n) = 1 + 2 + 3 + \cdots + n$. Which is true?

- [A] $S(n) \in \Theta(n)$.
- [B] $S(n) \in \Theta(n \log_2 n)$.
- [C] $S(n) \in \Theta(n^2)$.
- [D] $S(n) \in \Theta(2^n)$.
- [E] None of the above.

C

$S(n) = \frac{1}{2} n(n+1) \in \Theta(n^2)$

(5) Compute gcd(210, 385). That is, compute the greatest common divisor of 210 and 385.

- [A] 3.
- [B] 5.
- [C] 7.
- [D] 9.
- [E] None of the above.

E

$\gcd(m,n) = \gcd(\text{rem}(n,m), m)$

$\gcd(210, 385) = \gcd(175, 210)$

$= \gcd(35, 175)$

$= \gcd(0, 35)$

$= \boxed{35}$

(6) Compute the remainder when 6^{2025} is divided by 7.

[A] 1.

[B] 3.

[C] 5.

D [D] 6.

[E] None of the above.

$6 \equiv -1 \pmod 7$

$6^{2025} \equiv (-1)^{2025}$

$\equiv -1$

$\equiv 6.$

(7) A friendship network (simple graph) has degree sequence [4, 4, 3, 3, 2]. How many edges does it have?

[A] 7.

[B] 8.

B [C] 9.

[D] 10.

[E] None of the above, or not enough information.

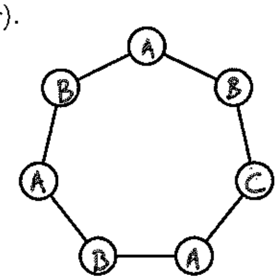
$\sum \text{degrees} = 4+4+3+3+2$

$= 16$

$= 2|E|$

$\rightarrow |E| = 8.$

(8) You wish to color the graph on the right so that linked vertices do not get the same color. What is the minimum number of colors needed (the chromatic number).



[A] 2

[B] 3

B [C] 4

[D] 5

[E] 6

(9) What is the minimum number of children that guarantees at least two have the first and last initial 'A'?

[A] 26

[B] 26 + 1

[C] 26^2

E [D] $26^2 + 1$

[E] None of the above.

Every one can have first initial B so not possible

(10) A race has 10 runners. In how many ways can the 10 runners finish the race?

[A] 10^{10}

[B] $\binom{10}{5}$

[C] $10 \times 9 \times 8$

D [D] 10!

[E] None of the above.

10! (Product rule: $10 \times 9 \times 8 \times \ldots \times 1$)

(11) You flip a coin. You then flip a second coin. You roll a die. You then roll a second die. How may possible outcomes of this experiment are there?

A 4.

B 12.

C 36.

D 144.

E None of the above.

D

Flip	Flip	Roll	Roll
c_1	c_2	d_1	d_2.
2	2	6	6

Product Rule

$2 \times 2 \times 6 \times 6 = 12 \times 12 = \boxed{144}$

(12) You have 10 different books. In how many ways can you give Alice 3 books and then give Bob 2 books?

A $\binom{10}{5}$.

B $10!/(2! \times 3! \times 5!)$.

C $10!/5!$.

D 10^5.

E None of the above.

B

① Choose 3 for Alice $\binom{10}{3}$

② Choose 2 for Bob $\binom{7}{2}$

Product Rule $\binom{10}{3} \times \binom{7}{2} = \frac{10!}{3!\cancel{7!}} \times \frac{\cancel{7!}}{5!2!} = \boxed{\frac{10!}{2!3!5!}}$

(13) How many 5 bit strings begin with 1 or have at least 3 ones?

A 5.

B 16.

C 21.

D 32.

E None of the above.

C

Begin 1 : $2^4 = 16$

≥ 3 ones : $\binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 10 + 5 + 1 = 16.$

Begin 1 AND ≥ 3 ones : $\underbrace{1\ x_1\ x_2\ x_3\ x_4}_{\text{At least 2 ones.}} = \underbrace{\binom{4}{2} + \binom{4}{3} + \binom{4}{4}}_{6 + 4 + 1 = 11}$

Inclusion-Exclusion : $16 + 16 - 11 = 32 - 11 = \boxed{21}$

(14) How many subsets of the set $\{1, 2, 3, 4, 5\}$ have size 2 or 3?

A 5.

B 10.

C 20.

D 32.

E None of the above.

C

$$\left.\begin{array}{l} \underline{\text{Size 2}} : \binom{5}{2} \to \frac{5!}{2!3!} = 10 \\ \underline{\text{Size 3}} : \binom{5}{3} \to \frac{5!}{3!2!} = 10 \end{array}\right\} 10 + 10 = \boxed{20}$$

(15) How many 10-bit strings contain 00 as a substring [Hint: count strings *not* containing 00 as a substring].

A $\binom{10}{8}$.

B 2^8.

C 880.

D 1024.

E None of the above.

C

$Q(n)$ = # n bit strings $\boxed{\text{not}}$ containing 00. Total Strings = 1024

n-bit string no 00 → starts 0 → starts 01 + n-2 bits, no 00 → $Q(n-2)$

n-bit string no 00 → starts 1 + (n-1) bits no 00 → $Q(n-1)$

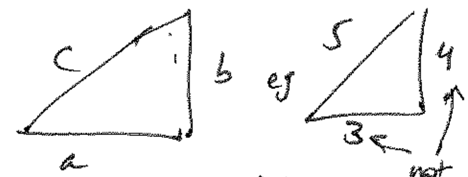
$\therefore Q(n) = Q(n-1) + Q(n-2)$ $Q(1) = 2$ $Q(2) = 3$.

We Want $1024 - 144 = \boxed{880}$

n	1	2	3	4	5	6	7	8	9	10
$Q(n)$	2	3	5	8	13	21	34	55	89	144

2 **Prove. In a right triangle with integer sides, the two shorter sides cannot both be odd.**

$$a^2+b^2=c^2$$



a, b, c are all integers. not both odd.

<u>Proof</u> by Contradiction

Suppose a, b are odd.

$\to$ $a = 2k+1$, $b = 2l+1$ $\to$ $(2k+1)^2 + (2l+1)^2 = c^2$

odd + odd = even

$\therefore$ c^2 is even

$\therefore$ c is even

$\therefore$ $c = 2m$.

$$(2k+1)^2 + (2l+1)^2 = (2m)^2$$

$$4k^2+4k+1 + 4l^2+4l+1 = 4m^2$$

$$4(k^2+k+l^2+l) + 2 = 4m^2$$

LHS is not div by 4
RHS is div. by 4 } $\boxed{\text{FISHY}}$ / Contradiction

$\square$

13: tinkered with several examples to understand.

20: General Idea Proof by Contradiction.
Made some progress

25: Correct Proof.

3 Prove *by induction*. $\sum_{i=1}^{n}(i+1)2^i = n \times 2^{n+1}$, for $n \geq 1$.

Proof by induction

① Base Case $n=1$ $\sum_{i=1}^{1}(i+1)2^i = 2 \cdot 2^1 = 4$

$n \times 2^{n+1} = 1 \times 2^2 = 4$ } ✓.

② Induction Step.

Assume $\sum_{i=1}^{n}(i+1)2^i = n \times 2^{n+1}$

Prove $\sum_{i=1}^{n+1}(i+1)2^i = (n+1)\, 2^{n+2}$

$$\sum_{i=1}^{n+1}(i+1)2^i = \sum_{i=1}^{n}(i+1)2^i + (n+2)\,2^{n+1}$$

$$= n\, 2^{n+1} + (n+2)\, 2^{n+1}$$

$$= 2^{n+1}(n + n + 2)$$

$$= 2^{n+1}(2n+2)$$

$$= (n+1)\, 2^{n+2} \quad \text{(factor out 2)}.$$

'13: Showed understanding + some tinkering

20: Base case plus some induction progress

25: Correct Proof.

4 Compute a formula for this sum, $S(n) = \sum_{i=0}^{n} \sum_{j=0}^{n} (2^i + 2^j)^2$.

$$S(n) = \sum_{i=0}^{n}\sum_{j=0}^{n} \left(2^i + 2^j\right)^2 = \sum_{i=0}^{n} \sum_{j=0}^{n} \left(2^{2i} + 2^{2j} + 2^{i+j} \cdot 2\right).$$

Inner Sum first

$$\sum_{j=0}^{n} 2^{2i} + 2^{2j} + 2 \cdot 2^{i+j} = \sum_{j=0}^{n} 2^{2i} + \sum_{j=0}^{n} 2^{2j} + 2 \cdot \sum_{j=0}^{n} 2^{i+j}$$ (Sum Rule).

$$= 2^{2i} \times (n+1) + \sum_{j=0}^{n} 4^j + 2 \cdot 2^i \sum_{j=0}^{n} 2^j$$ (Constant Rule)

$$\underbrace{= (1+n) 2^{2i} + \frac{4^{n+1}-1}{3} + 2 \cdot 2^i \left(2^{n+1}-1\right)}_{\text{Use this in outer Sum.}}$$ (Common Sums).

$$\sum_{i=0}^{n}\left((n+1)2^{2i} + \frac{4^{n+1}-1}{3} + 2 \cdot 2^i\left(2^{n+1}-1\right)\right) = (n+1)\sum_{i=0}^{n} 2^{2i} + \frac{4^{n+1}-1}{3}\sum_{i=0}^{n} 1 + 2\left(2^{n+1}-1\right)\sum_{i=0}^{n} 2^i$$

$$\underbrace{= (n+1) \underbrace{\sum_{i=0}^{n} 4^i}_{\left(\frac{4^{n+1}-1}{3}\right)} + (n+1)\left(\frac{4^{n+1}-1}{3}\right) + 2\left(2^{n+1}-1\right)\left(2^{n+1}-1\right)}_{\text{Common Sums.}}$$

$$= \boxed{\frac{2(n+1)\left(4^{n+1}-1\right)}{3} + 2\left(2^{n+1}-1\right)^2}.$$

13: Tinkered to show understanding

20: Made some progress with squaring and breaking up sum.

25: More or less correct proof. Uses right ideas. Algebra errors can be forgiven if small.

5 **Prove the binomial theorem *by induction* [Hint: Pascal's Identity]. For real numbers $x, y,$** and $n \geq 1$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

HARD

Proof By Induction

① Base Case: $(x+y)^1 = x+y$

$\sum_{i=0}^{1} \binom{1}{i} x^i y^{n-i} = x+y$ ✓.

② Assume $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

Prove $(x+y)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} x^i y^{n+1-i}$

$$(x+y)^{n+1} = (x+y)^n (x+y)$$

$$= \left(\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \right)(x+y)$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{i+1} y^{n-i} + \sum_{i=0}^{n} \binom{n}{i} x^i y^{n+1-i}$$

$$= \binom{n}{0} x^1 y^n + \binom{n}{1} x^2 y^{n-1} + \binom{n}{2} x^3 y^{n-2} + \dots \binom{n}{n} x^{n+1} y^0$$

$$+ \binom{n}{0} x^0 y^{n+1} + \binom{n}{1} x^1 y^n + \binom{n}{2} x^2 y^{n-1} + \dots \binom{n}{n} x^n y^1$$

$$= \binom{n}{0} x^0 y^{n+1} + \underbrace{\left(\binom{n}{0}+\binom{n}{1}\right)} x^1 y^n + \underbrace{\left(\binom{n}{1}+\binom{n}{2}\right)} x^2 y^{n-1} + \dots \underbrace{\left(\binom{n}{n-1}+\binom{n}{n}\right)} x^n y^1 + \binom{n}{n} x^{n+1} y^0$$

$$= \underbrace{\binom{n+1}{0}}_{\text{both are 1}} x^0 y^{n+1} + \underbrace{\binom{n+1}{1}}_{\text{Pascal}} x^1 y^n + \underbrace{\binom{n+1}{2}}_{\text{Pascal}} x^2 y^{n-1} + \dots \underbrace{\binom{n+1}{n}}_{\text{Pascal.}} x^n y^1 + \underbrace{\binom{n+1}{n+1}}_{\text{both are 1}} x^{n+1} y^0$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} x^i y^{n+1-i}$$

////

13: Basic Induction Infrastructure + Base Case.
20: Related $(x+y)^{n+1}$ to $(x+y)^n$ by multiplying sum $\times (x+y)$
25: Able to ~~produce~~ the logical steps and see where Pascal comes in. Mostly correct.