QUIZ 3: 90 Minutes

Last Name: Solutions

First Name: __________

RIN: __________

Section: __________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

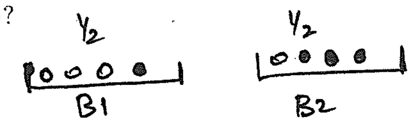
Circle at most one answer per question.
10 points for each correct answer

Total
100

1. The first 2 questions refer to the following experiment.

> There are two identical bags. One contains 3 white and 1 black ball; the other 1 white and 3 black balls. You pick a bag randomly (probability $\frac{1}{2}$ for each bag) and then randomly pick one of the balls in the bag (probability $\frac{1}{4}$ for each ball). You got a white ball. Let X be the number of white balls in the *other* bag. *(The* **information** *that you got a white ball is very important.)*

What is $\mathbb{E}[X]$ (expected value)?



$P[B1|W] = \frac{P[B1 \wedge W]}{P[W]}$, $P[W] = 1/2$, $= \frac{\frac{1}{2}\cdot 3/4}{1/2} = 3/4$

$P[B2|W] = 1/4$

$X = \begin{cases} 3 & \text{w. p. } 1/4 \\ 1 & \text{w. p. } 3/4 \end{cases}$

$E[X] = \frac{3}{4} + \frac{3}{4} = \frac{6}{4}$

[A] 1

[B] $\frac{6}{4}$

[C] $\frac{10}{4}$

[D] 2

[E] $\frac{5}{4}$

B

2. What is $Var(X)$ (variance)?

$E[X^2] = 9\cdot\frac{1}{4} + 1\cdot\frac{3}{4} = \frac{12}{4} = 3$

$Var(X) = E[X^2] - E[X]^2 = 3 - \frac{9}{4} = \frac{12-9}{4} = \frac{3}{4}.$

[A] $\frac{2}{4}$

[B] $\frac{3}{4}$

[C] 1

[D] $\frac{5}{4}$

[E] $\frac{6}{4}$

B

3. A game costs $\$x$ to play. You toss 4 fair coins. If you get *more* heads than tails, you win and get back $\$10 + x$ for a *profit* of $10. Otherwise, you lose and get nothing back, so your *loss* is $\$x$. What is an expression for your expected profit in dollars?

3 heads or 4 heads. $P[win] = \binom{4}{3}\frac{1}{2^4} + \binom{4}{4}\frac{1}{2^4} = \frac{4+1}{2^4} = \frac{5}{16}$

$E[Profit] = 10\cdot\frac{5}{16} + \frac{-x\cdot 11}{16} = \frac{50-11x}{16}.$

[A] $10 \times \frac{1}{2} - x \times \frac{1}{2}$

[B] $\frac{50-11x}{16}$

[C] $\frac{60-10x}{16}$

[D] $\frac{50-x}{16}$

[E] $\frac{60-x}{16}$

B

4. A Martian couple continues to have children until they have 2 males (not necessarily in a row). On Mars, males are twice as likely as females. Assume children are *independent.* Let X be the number of children this couple will have. What is $\mathbb{E}[X]$, the expected number of children this couple will have?

[A] 2

[B] 3

[C] 2.5

[D] 3.5

[E] 4

$P[\text{male}] = \frac{2}{3}$ $\quad P[\text{female}] = \frac{1}{3}.$

$X_1 =$ time to first male $\quad X_2 =$ time to second male.

$X = X_1 + X_2 \quad E[X] = E[X_1] + E[X_2] = \frac{3}{2} + \frac{3}{2} = 3.$

B.

5. You toss 5 independent fair coins. What is the probability that you will get 4 or more heads?

[A] $\binom{5}{4} \times \frac{1}{2^5}$

[B] $\frac{3}{16}$

[C] $\frac{5}{32}$

[D] $\frac{1}{4}$

[E] $\frac{9}{32}$

$$P[4 \text{ or more heads}] = P[4] + P[5] = \binom{5}{4}\frac{1}{2^5} + \binom{5}{5}\frac{1}{2^5} = \frac{5}{32} + \frac{1}{32} = \frac{6}{32} = \frac{3}{16}.$$

B.

6. Step 1: Toss 9 fair coins. Step 2: if you got more heads than tails in Step 1, toss 9 more coins and stop; if you get fewer heads than tails in Step 1, stop. Let X be the number of heads you toss. What is $\mathbb{E}[X]$?

[A] 6.25

[B] 6.75

[C] 7.25

[D] 9

[E] 8

$\underbrace{X_1 X_2 X_3 \ldots X_9}_{\text{Step 1.}} \quad \underbrace{Y_1 Y_2 \ldots Y_9}_{\text{Step 2.}}$

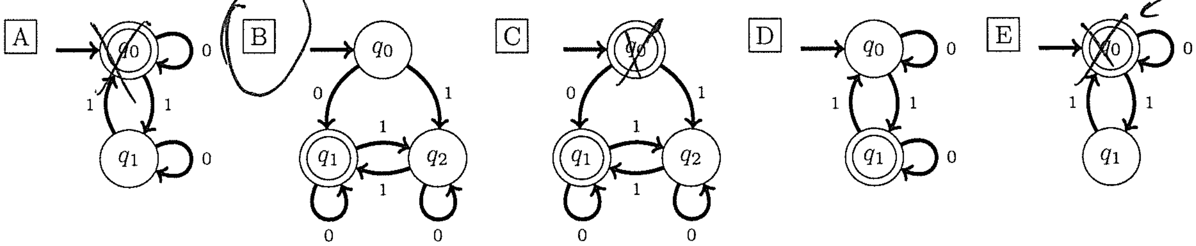
$X = X_1 + \ldots + X_9 + Y_1 + \ldots Y_9.$ $\quad X_i = 1$ w.p. $1/2$

$\frac{9}{2} + \frac{9}{4} = \frac{27}{4} = 6.75$ $\quad Y_i = 1$ w.p. $\frac{1}{2} \cdot \frac{1}{2}$

↑ prob > heads.

B.

7. Language $\mathcal{L}_1$ = {all *non-empty* strings in which the number of 1's is even}. Which finite automaton solves this problem, ~~i.e. the~~ YES-set (set of accepted strings) for the automaton is $\mathcal{L}_1$?



not accept empty string.

B.

8. Language $\mathcal{L}_2$ = {*all* strings in which the number of 1's is even} which CFG solves this problem - i.e., generates the strings in $\mathcal{L}_2$?

[A] $S \to \varepsilon \mid 0S \mid S0 \mid 11S \mid S11$ — can't get 101

[B] $S \to \varepsilon \mid 0S \mid S0 \mid 1S1$ — starts with or ends with 0 use $S \to 0S$ or $S \to S0$ otherwise use $S \to 1S1$

[C] $S \to \varepsilon \mid 0S \mid 11S$ — (can't start with 01)

[D] $S \to \varepsilon \mid 1S \mid S1 \mid 0S0$ — can generate 1 which is wrong!

[E] $S \to \varepsilon \mid 0 \mid 11 \mid SS$ — Can't get 101

B.

9. Which of the following is *countable*?

[A] The set of real numbers.

[B] A language (a possibly infinite set of *finite* strings).

[C] The set of all subsets of $\mathbb{N}$.

[D] The set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

[E] The set of all functions from $\mathbb{N}$ to $\mathbb{N}$.

B.

10. Which of the following is *not* a valid way to show that a set S is countable:

[A] Show an onto function from $\mathbb{N}$ to S. ✓

[B] Show a 1-to-1 function from $\mathbb{N}$ to S.

[C] Show a bijection from $\mathbb{N}$ to S. ✓

[D] Show there *does not exist* a 1-to-1 function from $\mathbb{N}$ to S. ✓

[E] Show a 1-to-1 function from S to $\mathbb{N}$. ✓

B.