QUIZ 2: 60 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer

You **MUST** show work to get full credit.

Total
150

1. A drawer has 10 red and 10 blue socks. What is the minimum number of socks must you pull out to *guarantee* getting a pair of the same color?

B

A 2.

B 3.

C 11.

D 12.

E None of the above.

3 socks → two must be in the same color bin.

2. A drawer has 10 red and 10 blue socks. What is the minimum number of socks you must pull out to *guarantee* getting a blue pair of socks?

D

A 2.

B 3.

C 11.

D 12.

E None of the above.

first 10 can be red → need two more → 12

3. How many numbers in the set $\{1, 2, 3, \ldots, 1000\}$ are divisible by 4 <u>*or*</u> 6.

E

A 250.

B 330.

C 375.

D 416.

E None of the above.

$A_4 = \{\text{div by } 4\}$ $\quad |A_4| = \lfloor \frac{1000}{4} \rfloor = 250$

$A_6 = \{\text{div by } 6\}$ $\quad |A_6| = \lfloor \frac{1000}{6} \rfloor = 166$

$A_{4\cap 6} = \{\text{div by 4 and 6}\} = \{\text{div by } 12\}$ $\quad |A_{4\cap 6}| = \lfloor \frac{1000}{12} \rfloor = 83$

$|A_4 \cup A_6| = |A_4| + |A_6| - |A_{4\cap B}| = 250 + 166 - 83 = 416 - 83 = \underline{333}$

4. 100 runners finish a race. We are interested in the order of the first 10 runners. How many possible top-10 finishes are there?

A.

A $\dfrac{100!}{90!}$.

B $\dfrac{100!}{10! \times 90!}$.

C 100^{10}.

D 10^{100}.

E None of the above.

$100 \times 99 \times \ldots \times 91 = \frac{100}{90!}$

5. 100 runners finish a race. The top-10 will form the State-team for the Nationals. How many possible state teams are there?

A $\frac{100!}{90!}$.

B $\frac{100!}{10! \times 90!}$.

C 100^{10}.

D 10^{100}.

E None of the above.

B

$\binom{100}{10} = \frac{100!}{90!\,10!}$

6. 10 end-of-season awards (sportsmanship, most improved, fittest, leadership, etc.) are given to 100 runners (the same runner may get more than one award). In how many ways can the 10 prizes be awarded?

A $\frac{100!}{90!}$.

B $\frac{100!}{10! \times 90!}$.

C 100^{10}.

D 10^{100}.

E None of the above.

C

100 ways to give each prize

→ product rule $\underbrace{100 \times 100 \times \cdots 100}_{10 \text{ times}} = 100^{10}$

7. You have 3 red, 3 green and 3 blue flags. In how many ways can you arrange the flags along the street for the parade?

A 9!.

B 3^9.

C $9!/(3!)^3$.

D 9^3.

E None of the above.

C

Choose 3 from 9 for red
Choose 3 from 6 for green
remaining 3 for blue

$\binom{9}{3} \times \binom{6}{3} \times \binom{3}{3} = \frac{9!}{3!\cancel{6!}} \times \frac{\cancel{6!}}{3!3!} \times \frac{\cancel{3!}}{\cancel{3!}0!}$

$= \frac{9!}{(3!)^3}$

8. In how many different ways can you place 10 identical rings on your 10 fingers? (All 10 rings can fit on one finger.)

A 10!.

B 10^{10}.

C $\binom{20}{10}$.

D $\binom{19}{9}$.

E None of the above.

D

- $x_1 + x_2 + \ldots x_{10} = 10$ $\quad x_i$ = # rings on finger i

→ $\binom{19}{9} = \binom{10+10-1}{10-1}$

- Ten "types" of rings [one for each finger]

How many ways to choose 10 candies from 10 types

$\binom{10+10-1}{10-1} = \binom{19}{9}$

9. A probability space has four outcomes $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with probabilities $P_1 = x$, $P_2 = 2x$, $P_3 = 3x$, $P_4 = 4x$. What is x?

[A] 0.1.

[B] 0.2.

[C] 0.3.

[D] 0.4.

[E] None of the above.

A

Probabilities must sum to 1

$x + 2x + 3x + 4x = 10x = 1 \longrightarrow x = \frac{1}{10} = 0.1$

10. Randomly pick a 10-bit sequence (independent bits and each bit is 1 with probability $\frac{1}{2}$). What is the probability that the sequence starts with 111?

[A] 2^{-3}.

[B] 2^{-5}.

[C] 2^{-7}.

[D] 2^{-10}.

[E] None of the above.

A

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ $= 2^{-3}$

1 1 1 any

11. A box has 6 fair coins and 4 two-headed coins ($\mathbb{P}[\mathrm{H}] = 1$ for two-headed coins). You picked a coin randomly, tossed it and got H. What is the probability you picked the fair coin, $\mathbb{P}[\text{fair coin} \mid \text{you got H}]$?

[A] $\frac{2}{5}$.

[B] $\frac{3}{7}$.

[C] $\frac{5}{10}$.

[D] $\frac{7}{10}$.

[E] None of the above.

B

$P[\text{fair} / H] = \frac{P[\text{fair} \cap H]}{P[H]} = \frac{P[H/\text{fair}]P[\text{fair}]}{P[H]}$

$P[H/\text{fair}] = \frac{1}{2}$ $\quad P[\text{fair}] = \frac{6}{10}.$ $\quad \frac{1}{2} \times \frac{6}{10} = \frac{3}{10}$

$P[H] = \underset{\frac{1}{2}}{P[H/\text{fair}]}\,\underset{\frac{6}{10}}{P[\text{fair}]} + \underset{1}{P[H/\text{biased}]}\,\underset{\frac{4}{10}}{P[\text{biased}]} = \frac{7}{10}.$

$\frac{3/10}{7/10} = \frac{3}{7}.$

12. Toss 20 fair coins independently. All coins that came up heads are *re-flipped* (the coins that came up tails are *not* re-flipped). At the end, what is the probability that 10 heads in total are flipped?

[A] 0.5.

[B] $\binom{20}{10} \times \left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{10}$.

[C] $\binom{20}{10} \times \left(\frac{1}{3}\right)^{10} \times \left(\frac{2}{3}\right)^{10}$.

[D] $\binom{20}{10} \times \left(\frac{1}{4}\right)^{10} \times \left(\frac{3}{4}\right)^{10}$.

[E] None of the above.

D

~~P[H] ma~~ H means both tosses are H.

P[H] for a coin at the end $= \frac{1}{4}$

Binomial distribution: $n = 20$ $\quad p = \frac{1}{4}$ $\quad k = 10$

$\binom{n}{k} (p)^k (1-p)^{n-k} = \binom{20}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{10}.$

13. You roll a *loaded* 6-sided die (values 1,...,6). The probability of an even roll is twice the probability of an odd roll. What is the expected value of a single roll of this loaded die?

A $3\frac{1}{3}$.

B $3\frac{1}{2}$.

C $3\frac{2}{3}$.

D 4.

E None of the above.

C

1	2	3	4	5	6
x	$2x$	x	$2x$	x	$2x$
$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$

$9x = 1 \to x = \frac{1}{9}$

$$E[X] = \frac{1}{9}\cdot 1 + \frac{2}{9}\cdot 2 + \frac{1}{9}\cdot 3 + \frac{2}{9}\cdot 4 + \frac{1}{9}\cdot 5 + \frac{2}{9}\cdot 6$$

$$= \frac{1}{9}[1+4+3+8+5+12] = \frac{33}{9} = \frac{11}{3} = 3\frac{2}{3}$$

14. A box has 6 fair coins and 4 two-headed coins. You pick a coin randomly and make 10 independent flips. Let **X** be the number of heads you get. What is $\mathbb{E}[\mathbf{X}]$? [Hint: Total Expectation]

A 5.

B 6.

C 7.

D 8.

E None of the above.

C

$$E[X] = \underbrace{E[X \mid fair]}_{10\times\frac{1}{2}} \underbrace{P[fair]}_{\frac{6}{10}} + \underbrace{E[X \mid biased]}_{10\times 1} \underbrace{P[biased]}_{\frac{4}{10}}$$

$$= 5 \times \frac{3}{5} + 10 \times \frac{2}{5} = 3 + 4 = 7.$$

15. [Hard] A box has 1024 fair coins and 1 two-headed coin. You picked a random coin, flipped it 10 times and all 10 flips were H. You now keep flipping the *same* coin you picked until you flip a H. Let **X** be the number of additional flips you make. What is $\mathbb{E}[\mathbf{X}]$, the expected value of **X**?

A 1.

B $1\frac{1}{2}$.

C 2.

D $2\frac{1}{2}$.

E None of the above.

B

$$P[fair \mid 10H] = \frac{P[10H \cap fair]}{P[10H]} = \frac{\frac{1024}{1025} \times \frac{1}{2^{10}}}{P[10H]} = \frac{\frac{1}{1025}}{P[10H]}.$$

$$P[biased \mid 10H] = \frac{P[10H \cap biased]}{P[10H]} = \frac{\frac{1}{1025} \times 1}{P[10H]} = \frac{\frac{1}{1025}}{P[10H]}.$$

$P[fair \mid 10H] = P[biased \mid 10H] \to$ both are $\frac{1}{2}$.

$$E[X] = E[X \mid fair] P[fair] + E[X \mid biased] P[biased]$$

$$= 2 \times \frac{1}{2} + 1 \times \frac{1}{2}$$

$$= 1\frac{1}{2}$$