QUIZ 3: 60 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer

You **MUST** show work to get full credit.

Total
150

1. Which of the following describes the expected value of a random variable $\mathbf{X}$?

Expected value may not occur. (eg dice, Exp Value is $3\frac{1}{2}$).

E

A It is the typical observed value of $\mathbf{X}$ in an experiment. ✗

B It is the most likely observed value of $\mathbf{X}$ in an experiment. ✗

C It is one of the possible observed values of $\mathbf{X}$ in an experiment. ✗

D It is the maximum value of $\mathbf{X}$ that can be observed in an experiment. ✗

E None of the above.

2. For a random variable $\mathbf{X}$, what does the standard deviation $\sigma(\mathbf{X})$ measure?

$\sigma(x)$ measures deviations from the mean.

C

A The average value of $\mathbf{X}$ you will observe if you ran the experiment many times. ✗

B The number of times you run the experiment (on average) before you observe the value $\mathbb{E}[\mathbf{X}]$. ✗

C The size of the deviation between the observed value of $\mathbf{X}$ and the expected value $\mathbb{E}[\mathbf{X}]$.

D The probability that $\mathbf{X}$ will be larger than its expected value $\mathbb{E}[\mathbf{X}]$.

E The number of possible values of $\mathbf{X}$.

3. A real valued $\mathbf{X}$ has expectation $\mathbb{E}[\mathbf{X}] = \mu$. Which is *not* a valid formula for the variance $\sigma^2(\mathbf{X})$?

E

A $\mathbb{E}[(\mathbf{X}-\mu)^2]$. ✓ ← definition

B $\mathbb{E}[\mathbf{X}^2] - 2\mu\,\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{X}]^2$. ✓

C $\mathbb{E}[\mathbf{X}^2] - \mu^2$. ✓

D $\mathbb{E}[\,|\mathbf{X}|^2\,] - \mu^2$. ✓

E They are all valid.

$$E[(x-\mu)^2] = E[x^2 - 2\mu x + \mu^2] = E[x^2] - 2\mu E[x] + \mu^2 = E[x^2] - 2\mu\cdot\mu + \mu^2 = E[x^2] - \mu^2$$

$$|x|^2 = x^2$$

4. A class has 10 students. Each student is given a random number in $\{1, 2, 3, \ldots, 10\}$. The score $\mathbf{X}$ for the class is now computed as follows. For every pair of students whose numbers match, the number is added *once* to the score. For example, if the numbers given to the students are $\{1, 1, 1, 2, 3, 4, 5, 8, 10, 10\}$, then the score $\mathbf{X} = 13$. What is an approximate value for $\mathbb{E}[\mathbf{X}]$? *[Hint: Linearity of expected value.]*

B

A $\mathbb{E}[\mathbf{X}] \approx 10$.

B $\mathbb{E}[\mathbf{X}] \approx 25$.

C $\mathbb{E}[\mathbf{X}] \approx 55$.

D $\mathbb{E}[\mathbf{X}] \approx 105$.

E $\mathbb{E}[\mathbf{X}] \approx 155$.

let X_{ij} be the score from pair of students (i,j)

$$X = \sum_{i<j} X_{ij} \qquad E[X_{ij}] = \sum_{x=1}^{10} \underbrace{P(i \text{ gets } x \text{ AND } j \text{ gets } x)}_{\frac{1}{100}}\cdot x = \frac{1}{100}\sum_{x=1}^{10} x = \frac{1}{100}\cdot\frac{10\times 11}{2}$$

$$E[X] = \sum_{i<j} E[X_{ij}] = \frac{1}{100}\cdot\frac{10\times 11}{2}\cdot\underbrace{\sum_{i<j} 1}_{\#\text{ of pairs}} \qquad = \frac{10\times 9}{2}$$

$$= \frac{1}{100}\times\frac{10\times 11}{2}\times\frac{10\times 9}{2} = \frac{99}{4} \approx \underline{\underline{25}}$$

5. A random variable **X** has PDF shown on the right. Compute $\mathbb{E}[\mathbf{X}]$ (expectation) and $\sigma^2(\mathbf{X})$ (variance).

$\mathbf{X}$	-2	-1	0	1	2
$P_{\mathbf{X}}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

A

- [A] $\mathbb{E}[\mathbf{X}] = 0$ $\quad \sigma^2(\mathbf{X}) = 2$
- [B] $\mathbb{E}[\mathbf{X}] = 0$ $\quad \sigma^2(\mathbf{X}) = 4$
- [C] $\mathbb{E}[\mathbf{X}] = 1$ $\quad \sigma^2(\mathbf{X}) = 4$
- [D] $\mathbb{E}[\mathbf{X}] = 1$ $\quad \sigma^2(\mathbf{X}) = 8$
- [E] None of the above.

Symmetric $\rightarrow E[x] = 0$

$$\sigma^2(x) = E[x^2] - \underset{\uparrow \atop 0}{E[x]}^2$$
$$= \frac{1}{5}\cdot(-2)^2 + \frac{1}{5}\cdot(-1)^2 + \frac{1}{5}\cdot 0^2 + \frac{1}{5}\cdot 1^2 + \frac{1}{5}\cdot 2^2$$
$$= \frac{10}{5} = 2.$$

6. For the random variable **X** in Problem 5 above, let $\mathbf{Y} = 2\mathbf{X} + 1$. Compute $\mathbb{E}[\mathbf{Y}]$ and $\sigma^2(\mathbf{Y})$.

D

- [A] $\mathbb{E}[\mathbf{Y}] = 0$ $\quad \sigma^2(\mathbf{Y}) = 2$
- [B] $\mathbb{E}[\mathbf{Y}] = 0$ $\quad \sigma^2(\mathbf{Y}) = 4$
- [C] $\mathbb{E}[\mathbf{Y}] = 1$ $\quad \sigma^2(\mathbf{Y}) = 4$
- [D] $\mathbb{E}[\mathbf{Y}] = 1$ $\quad \sigma^2(\mathbf{Y}) = 8$
- [E] None of the above.

$$E[Y] = 2\underset{\uparrow \atop 0}{E[x]} + 1 = 1$$

$\sigma^2(Y) = \sigma^2(2x + 1)$ ← adding constant does not change σ^2

$= \sigma^2(2x)$

$= 4\sigma^2(x)$ ← linear term squares

$= 4 \times 2 = 8$

7. **[Hard]** A Martian couple continues to have children until they have 2 males *in a row*. On Mars, *males are twice as likely as females.* Assume children are *independent*. Let **X** be the number of children this couple will have. Compute $\mathbb{E}[\mathbf{X}]$, the expected number of children this couple will have.

B

- [A] $2\frac{1}{4}$.
- [B] $3\frac{3}{4}$.
- [C] 6.
- [D] 12.
- [E] None of the above.

$$E[x] = \underset{\uparrow \atop 2}{E[x|BB]}\,\underset{\uparrow \atop \frac{2}{3}\times\frac{2}{3}}{P[BB]} + \underset{\uparrow \atop E[x]+2}{E[x|BG]}\,\underset{\uparrow \atop \frac{2}{3}\times\frac{1}{3}}{P[BG]} + \underset{\uparrow \atop E[x]+1}{E[x|G]}\,\underset{\uparrow \atop \frac{1}{3}}{P[G]}.$$

$P[B] = \frac{2}{3}$ $\rightarrow$ $E[x] = \frac{8}{9} + \frac{2}{9}E[x] + \frac{4}{9} + \frac{3}{9}E[x] + \frac{3}{9}$

$= \frac{15}{9} + \frac{5}{9}E[x].$

$\rightarrow$ $\frac{4}{9}E[x] = \frac{15}{9}$ $\rightarrow$ $E[x] = \frac{15}{4} = 3\frac{3}{4}.$

8. Which (if any) of the following sets *do not* have the same cardinality as $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$?

A

infinite

- [A] $\{0, 1, 2, 3, 4, 5\}$. ← finite
- [B] The rationals, $\mathbb{Q} = \{\frac{z}{n} \mid z \in \mathbb{Z}, n \in \mathbb{N}\}$.
- [C] The set of valid C++ programs.
- [D] The set of all possible Turing Machines.
- [E] They all have the same cardinality as $\mathbb{N}$.

9. Which (if any) of the following sets is **not** *countable*?

E

- [A] $\{0, 1, 2, 3, 4, 5\}$. ← finite ✓
- [B] The rationals, $\mathbb{Q} = \{\frac{z}{n} \mid z \in \mathbb{Z}, n \in \mathbb{N}\}$. ← listable. ✓
- [C] The set of valid C++ programs. ← subset of finite binary strings ✓
- [D] The set of all possible Turing Machines. ← subset of finite binary strings ✓
- [E] They are all countable.

10. Which (if any) is *not* a valid way to prove that a set S is countable?

E

- [A] Show an injection exists from S to $\mathbb{N}$. $|S| \leq |\mathbb{N}|$ ✓
- [B] Show a 1-to-1 function exists from S to $\mathbb{N}$. ← same as injection
- [C] Show a surjection exists from $\mathbb{N}$ to S. $|\mathbb{N}| \geq |S|$ ✓
- [D] Show that S is finite. finite is countable ✓
- [E] They are all valid ways to show S is countable.

11. Which of the following strings is *not* in the language described by the regular expression $\{0, 10\}^*$?

D

- [A] ε. ε
- [B] 010010. $0 \cdot 10 \cdot 0 \cdot 10$ ✓
- [C] 100100. $10 \cdot 0 \cdot 10 \cdot 0$ ✓
- [D] 010110. ×
- [E] They are all in the language.

$\varepsilon, 0, 10, 00, 010, 100, 1010, \ldots$

cannot get two 1's in a row.

12. Which computing problem (if any) *cannot* be solved by a DFA (deterministic finite automata)?

Suppose an automaton exists with k states consider 0^k
Automaton must go through the same state for 0^i + 0^j $i<j$
Consider $0^i 1^{i+1}$ $0^j 1^{i+1}$
accept → accept contradiction

D.

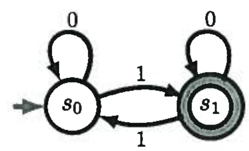
- [A] $\mathcal{L} = \{$strings with at least one 1$\}$. ✓
- [B] $\mathcal{L} = \{(01)^{\bullet n} \mid n \geq 0\}$. ✓
- [C] $\mathcal{L} = \{$strings that end with 01$\}$. ✓
- [D] $\mathcal{L} = \{$strings with *more* 1s than 0s$\}$.
- [E] They can each be solved by some DFA.

13. Which problem (if any) *cannot* be solved by a CFG (context free grammar)?

E

- [A] $\mathcal{L} = \{$strings with at least one 1$\}$. ← DFA ✓
- [B] $\mathcal{L} = \{(01)^{\bullet n} \mid n \geq 0\}$. ← DFA ✓
- [C] $\mathcal{L} = \{$strings that end with 01$\}$. ← DFA ✓
- [D] $\mathcal{L} = \{$strings with *more* 1s than 0s$\}$. ✓ ← $S \to AS \mid 1A \mid 1S$
  $A \to \varepsilon \mid 0A1A \mid 1A0A$.
- [E] They can each be solved by some CFG.

14. The DFA on the right solves a computing problem defined by its (YES)-set (the language it accepts). The accept state is s_1. What is a regular expression for this computing problem?



odd # of 1's

D

- [A] $\{0,1\}^*$. ← all strings
- [B] $\{0,1\}^* \bullet 1$. ← strings ending in 1
- [C] $\{0\}^* \bullet 1 \bullet \{\{0\}^* \bullet 1 \bullet \{0\}^* \bullet 10\}^*$ ← can only end in 0 if more than one 1
- [D] $\{0\}^* \bullet 1 \bullet \{\{0\}^* \bullet 1 \bullet \{0\}^* \bullet 1\}^* \bullet \{0\}^*$
- [E] None of the above.

$0^* \cdot 1 \cdot \{\{0\}^* \cdot 1 \cdot \{0\}^* \cdot 1\}^* \cdot 0^*$
first 1
string with 2 1's ending in 1
even # of 1s
end in unknown # of zeros.

15. Rank deterministic finite automata (DFA), context free grammars (CFG), which are related to pushdown automata, and Turing Machines (TM) in order of how powerful they are. (For example, $DFA > CFG$ if DFAs can solve more problems that CFGs; $DFA = CFG$ if DFAs and CFGs can solve the same problems; $DFA < CFG$ if DFAs can solve fewer problems that CFGs.

- [A] $DFA > CFG > TM$
- [B] $DFA = CFG > TM$
- [C] $DFA = CFG = TM$
- [D] $DFA = CFG < TM$
- [E] $DFA < CFG < TM$

DFA < CFG < TM

E