QUIZ 1: 60 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

Total
200

INSTRUCTIONS

1. This is a **closed book** test. No electronics, books, notes, internet, etc.
2. The test will become available in Submitty at 8am on the test date.
3. Your PDF is due in Submitty by 2pm.
4. By submitting the test you attest that:
   – the work is entirely your own.
   – you obeyed the time limits of the exam.
5. Your submission *must* be typed and submitted as a PDF file.
6. The first page should list your twenty answers, something like:

(1)	A
(2)	B
(3)	C
(4)	D
⋮	
(20)	A

7. The *second* page onward *must* show your work for *every* answer, e.g.:

(1)	Because x is even
(2)	Because $\sqrt{2}$ is irrational.
(3)	Number of links is $$1 + 2 + \cdots + 10 = 55$$
⋮	
(20)	Because we proved in class that $\ell = n - 1$

   – Some problems may be "easy", so give a one line justification
   – Some problems may require a detailed reasoning.
8. **If you don't show correct work, you won't get credit.**
9. Be careful. This is multiple choice.
   – Correct answers get 10 points.
   – Wrong answers or correct answers with no justification get 0.
10. Submit with plenty of time to spare. A late test won't be accepted.
    – We won't accept submissions that are even 1 second late.

1. Jodie asks John to solve $x^2 - a = 0$ and find x as a rational number. Which is true?

$x = \sqrt{a}$

$a = 2 \rightarrow x = \sqrt{2}$ irrational

$a = 4 \rightarrow x = 2$ rational.

$\boxed{A}$ $\forall a \in \mathbb{N}$: John can find a rational solution x. ×

$\boxed{B}$ $\forall a \in \mathbb{N}$: John cannot find a rational solution x. ×

$\boxed{C}$ $\forall a \in \mathbb{Z}$: John can find a rational solution x. ×

E

$\boxed{D}$ $\forall a \in \mathbb{Z}$: John cannot find a rational solution x. ×

$\boxed{E}$ None of the above.

2. The set $S = \{4, 16, 64, 256, 1024, \ldots\}$. Which of these definitions using a variable could be S?

$S = \{2^2, 2^4, 2^6, 2^8, 2^{10}, \ldots\}$

2^{2k} $k \in \mathbb{N}$

$\boxed{A}$ $S = \{n | n = 2^k, \text{ for } k \in \mathbb{N}\}$.

$\boxed{B}$ $S = \{n | n = 4^{1+k(k-1)/2}, \text{ for } k \in \mathbb{N}\}$.

$\boxed{C}$ $S = \{n | n = 2 \times 2^k, \text{ for } k \in \mathbb{N}\}$.

D

$\boxed{D}$ $S = \{x | x = 2^{2k}, \text{ for } k \in \mathbb{N}\}$.

$\boxed{E}$ None of the above.

3. $A = \{\text{positive multiples of } 2\}$ and $B = \{\text{positive multiples of } 3\}$. Which element is not in $\overline{A \cap B}$?

not in $\overline{A \cap B}$ means in $A \cap B$

∴ must be multiple of 2 and 3

$\boxed{A}$ 4. ×

$\boxed{B}$ 8. ×

$\boxed{C}$ 12. ✓

C

$\boxed{D}$ 16.

$\boxed{E}$ None of the above.

4. An integer $n \in \mathbb{Z}$ has a square that is divisible by 3, that is 3 divides n^2. Which claim *must be* true?

3 divides $n^2 \Longleftrightarrow$ 3 divides n

(proved in text).

$\boxed{A}$ n is odd.

$\boxed{B}$ n is even.

$\boxed{C}$ n is positive.

D

$\boxed{D}$ n is divisible by 3.

$\boxed{E}$ None of the above claims must be true.

5. If it rains on a day, then it rains the next day. Today it didn't rain. Which is true?

r = rain yesterday

p = rain today

q = rain tomorrow

we have $\neg p$

$r \rightarrow p$ $\quad p \rightarrow q$

$\neg p$ means can conclude $\neg r$

$\boxed{A}$ It will rain tomorrow.

$\boxed{B}$ It will not rain tomorrow.

$\boxed{C}$ It did rain yesterday.

D

$\boxed{D}$ It did not rain yesterday.

$\boxed{E}$ None of the above.

6. Which method would succeed in *proving* $p \to (q \vee r)$?

A

[A] You assumed p is true and showed q is true.

[B] You assumed q is false and showed p is false.

[C] You showed that p is true and that q is false.

[D] You showed that p is true and that both q and r are false.

[E] None of the above.

q is true means $q \vee r$ is true
∴ p is T means $q \vee r$ is true and $p \to (q \vee r)$ is proved.

7. Which method would succeed in *disproving* $p \to (q \vee r)$?

[A] You assumed p is true and showed q is true.

[B] You assumed q is false and showed p is false.

[C] You showed that p is true and that q is false.

D

[D] You showed that p is true and that both q and r are false.

[E] None of the above.

Counter example requires
p is T
$(q \vee r)$ is F
or q is F and r is F

8. Determine true or false for the claim $\forall n \in \mathbb{Z} : (n > n+1) \to (n+1 > n+2)$.

[A] This is not a valid proposition which is either true or false.

[B] True for $n < 0$ and false otherwise.

[C] True for $n = 0$ and false otherwise.

E

[D] False.

[E] True.

$n > n+1$ is always F ∴ $(n > n+1) \to$ (anything) is T.

Also Assume $n > n+1$
then $n+1 > n+1+1 = n+2$ } Direct proof.

9. What method of proof would you use to *prove* that you cannot choose $a, b \in \mathbb{Z}$ so that $a^2 - 4b = 2$?

[A] Direct proof

[B] Contraposition proof.

[C] Proof by induction.

D

[D] Proof by contradiction.

[E] None of the above.

Proved in text by contradiction

10. What method would you use to *prove* that $n^3 \leq 2^n$ for *all* $n \geq 10$?

[A] Direct proof

[B] Contraposition proof.

[C] Show that the formula is true for $n = 1$ up to $n = 1000$.

D

[D] Proof by induction.

[E] Proof by contradiction. → also ok if justified with well-ordering proof.

E accepted with wellordering justification

Proved in text by induction

11. We wish to prove $P(n)$ for all $n \geq 10$. Which method accomplishes this?

$P(n) \to P(n+2)$
$\underbrace{\quad}_{2}$ base cases
(2-leaping induction).
Base cases must be 10, 11

- [A] Prove base case $P(1)$ and prove $P(n) \to P(n+2)$ for all $n \geq 10$. ✗
- [B] Prove base cases $P(1), P(2)$ and prove $P(n) \to P(n+2)$ for all $n \geq 10$.
- [C] Prove base case $P(10)$ and prove $P(n) \to P(n+2)$ for all $n \geq 10$. ✗
- D. [D] Prove base cases $P(10), P(11)$ and prove $P(n) \to P(n+2)$ for all $n \geq 10$. ✓
- [E] None of the above methods works.

$\boxed{P(10)}\ \boxed{P(11)}\ P(12)\ P(13)\ P(14) \ldots$

12. For $x, y \in \mathbb{Z}$, which statement is *not necassarily* a contradiction? (That is, which could be true?)

- [A] $x + 0 > x + 1$. contradiction
- [B] $x \geq y$ AND $x < y$ Contradiction
- [C] $x^2 \geq y^2$ AND $|x| < |y|$ $|x| < |y| \to$ ~~[illegible]~~ $x^2 < y^2$ $\therefore$ contradiction.
- D [D] $x^2 + y^2 \leq 1$ $x = 0,\ y = 0 \to x^2 + y^2 \leq 1$ not contradiction
- [E] They are all contradictions.

13. Consider the predicate $P(n) : n^2 \leq 2^n$. Which claim is true? TINKER

- [A] $P(n)$ is true for at most finite number of $n \in \mathbb{N}$. ✗
- [B] $P(n)$ is true for *all* $n \in \mathbb{N}$. ✗
- C [C] $P(n)$ is true for *all* even $n \in N$. ✓
- [D] $P(n)$ is true for *all* odd $n \in N$. ✗
- [E] None of the above claims is true.

n	1	2	3	4	5	6	7	8	9
n^2	1	4	9	16	25	36	49	64	81
2^n	2	4	8	16	32	64	128	256	512
	✓	✓	✗	✓	✓	✓	✓	✓	✓

$P(2)$ and $\underbrace{P(n) \text{ for all } n \geq 4}_{\text{Proved in text by induction}}$

14. Consider the predicate $P(n) : 8$ divides $n^2 - 1$. Which claim is true? TINKER.

- [A] $P(n)$ is true for at most finite number of $n \in \mathbb{N}$ ✗
- [B] $P(n)$ is true for *all* $n \in \mathbb{N}$. ✗
- [C] $P(n)$ is true for *all* even $n \in N$. ✗
- D. [D] $P(n)$ is true for *all* odd $n \in N$. ✓
- [E] None of the above claims is true.

n	1	2	3	4	5	6	7	8	9	10.
n^2-1	0	3	8	15	24	35	48	63	80	99
	✓	✗	✓	✗	✓	✗	✓	✗	✓	

all odd n.

Can prove by induction or observe:

$n^2 - 1 = \underbrace{(n-1)(n+1)}$ ← Product of consecutive evens when n is odd.

$2k(2k+2) = 4\underbrace{k(k+1)}_{\text{even}}$ $\therefore$ multiple of 8.

15. Consider the predicate $P(n) : 1^2 + 2^2 + 3^2 + \cdots + n^2 > n^3/3$. Which claim is true? TINKER.

- [A] $P(n)$ is true for at most finite number of $n \in \mathbb{N}$.
- B [B] $P(n)$ is true for *all* $n \in \mathbb{N}$.
- [C] $P(n)$ is true for *all* even $n \in N$.
- [D] $P(n)$ is true for *all* odd $n \in N$.
- [E] None of the above claims is true.

n	1	2	3
Sum	1	5	14
$n^3/3$	1/3	8/3	27/3
	✓	✓	✓

← looks like all n.

Proof By induction

$P(n) \to P(n+1)$: Assume $1^2 + 2^2 + \ldots + n^2 > n^3/3$

$$1 + 2^2 + \cdots + n^2 + (n+1)^2 > \frac{n^3}{3} + (n+1)^2$$
$$= \frac{n^3 + 3n^2 + 6n + 3}{3}$$
$$> \frac{n^3 + 3n^2 + 3n + 1}{3} = \frac{(n+1)^3}{3} \quad \blacksquare$$

16. You wish to make postage n cents with 5-cent and 6-cent stamps. For which $n \in \mathbb{N}$ can you do it? TINKER

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	✓	✓	×	×	×	✓	✓	✓	×	×	✓	✓	✓	✓	×	✓	✓	✓	✓	✓

5555 5556 ~~5566~~ 5566 6666

5 in a row

and $P(n) \to P(n+5)$

$\therefore \forall n \geq 20$ is possible.

D

[A] All postages $n \geq 5$ cents.

[B] All postages $n \geq 10$ cents.

[C] All postages $n \geq 15$ cents.

[D] All postages $n \geq 20$ cents.

[E] None of the above.

17. $A_0 = 0$ and for $n > 0$, $A_n = n^2 + A_{n-2}$. What is A_6?

[A] It cannot be computed because this recurrence has only one base case.

[B] $A_6 = 12$.

[C] $A_6 = 52$.

[D] $A_6 = 56$.

[E] None of the above.

$A_6 = 36 + A_4 = 36 + 16 + A_2 = 36 + 16 + 4 + \cancel{A_0}^{0} =$

$= 36 + 20 = 56.$

D

18. $f(1) = 1; f(2) = 1$ and for $n > 2$, $f(n) = n + f(n-3)$. For which $n \in \mathbb{N}$ can $f(n)$ be computed?

[A] *All* $n \in \mathbb{N}$.

[B] *All* $n \in \mathbb{N}$ which are even.

[C] *All* $n \in \mathbb{N}$ which are multiples of 3.

[D] *All* $n \in \mathbb{N}$ which are not multiples of 3.

[E] None of the above.

$\boxed{f(1)}$ $\boxed{f(2)}$ $f(3)$ $f(4)$ $f(5)$ $f(6)$ $f(7)$ $f(8)$ $f(9)$ $f(10)$ $f(11)$ $f(12)$

multiples of 3 cant be done

D

19. Rooted binary trees (RBTs) are recursively defined below. How many RBTs have 4 vertices and 2 links?

links = vertices − 1

must have 3 links → NOT POSSIBLE

A

[A] 0

[B] 5

[C] 14

[D] 42

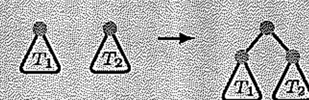
[E] 132

Recursive Definition of RBT

① The empty tree ε is an RBT.

② If T_1, T_2 are disjoint RBTs with roots r_1 and r_2, then linking r_1 and r_2 to a *new* root r gives a new RBT with root r.

③ Nothing else is an RBT.



20. T_1 and T_2 are disjoint RBTs. RBT T_1 has 8 vertices and 7 links. RBT T_2 has 4 vertices and 3 links. Using the constructor for RBT, you get a child RBT T. How many vertices and links does T have?

[A] 12 vertices and 10 links

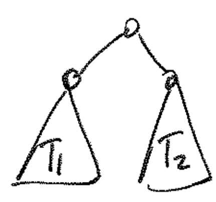
[B] 12 vertices and 11 links

[C] 13 vertices and 11 links

[D] 13 vertices and 12 links

[E] None of the above, or we can't say for sure.

D.



Vertices → vertices + 1 (total)

links → links + 2 (total)

$\therefore$ $8 + 4 \to 8 + 4 + 1 = 13$ Vertices

$7 + 3 \to 7 + 3 + 2 = 12$ links

SCRATCH