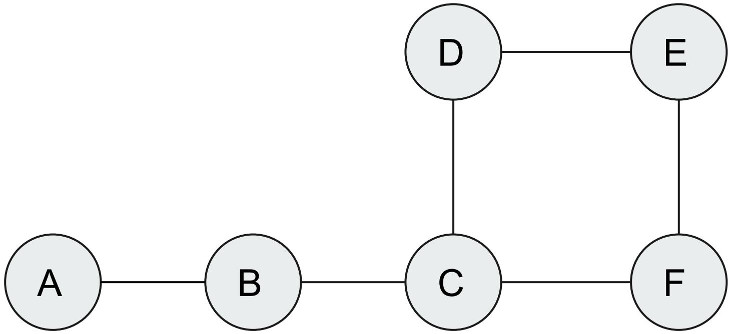
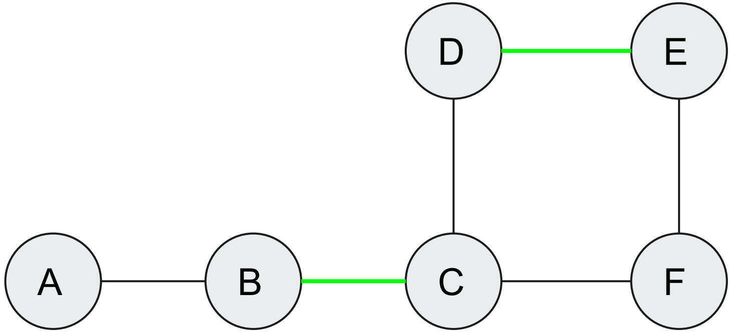
Graph Matchings

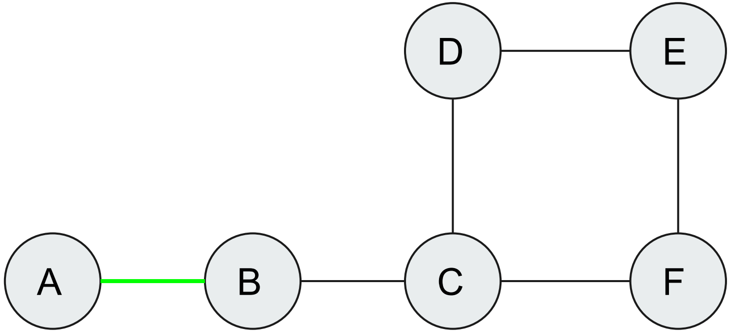
Matching

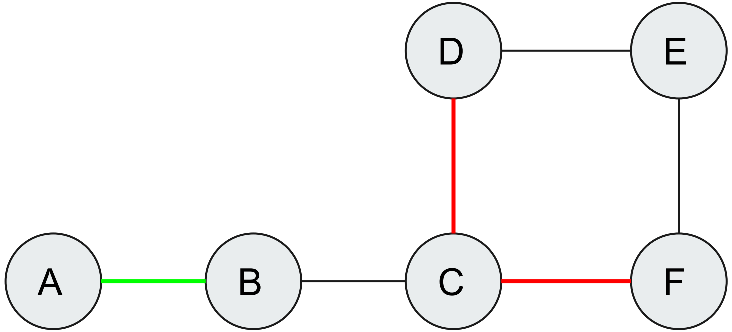
- A **matching** M in a graph G is a set of non-loop edges with no shared endpoints.



Matching

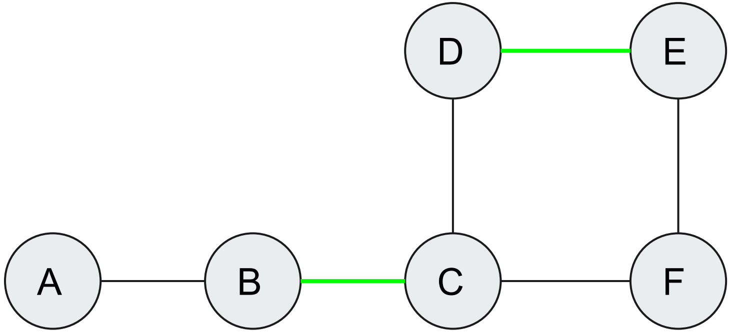






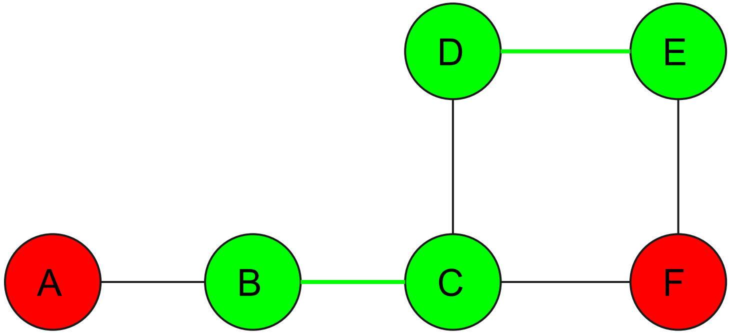
Matching

- A vertex is **saturated** if it is incident to *M*
- A vertex is **unsaturated** if it is not incident to *M*



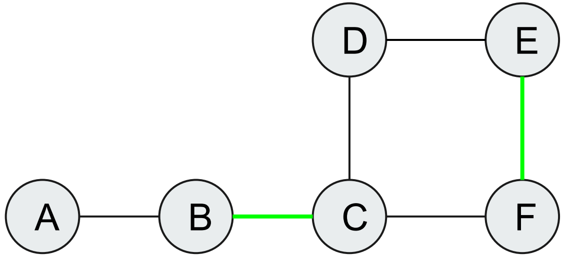
Matching

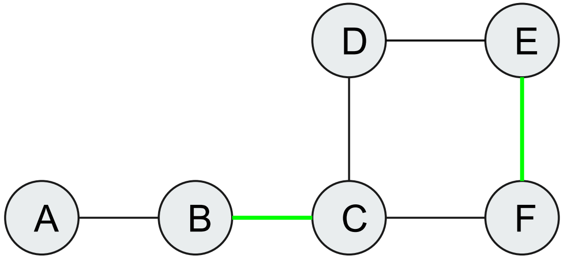
- A vertex is **saturated** if it is incident to *M*
- A vertex is **unsaturated** if it is not incident to *M*
- A **perfect matching** has all vertices saturated

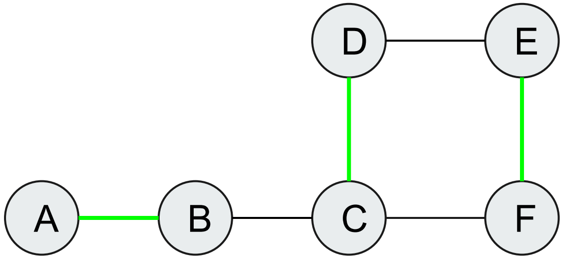


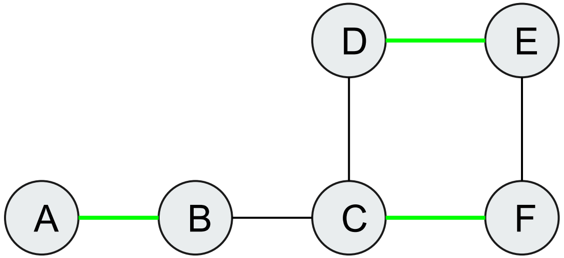
Maximal Matching

- A **maximal matching** is a matching that can't have any other edges added to it.
- Very easy to create



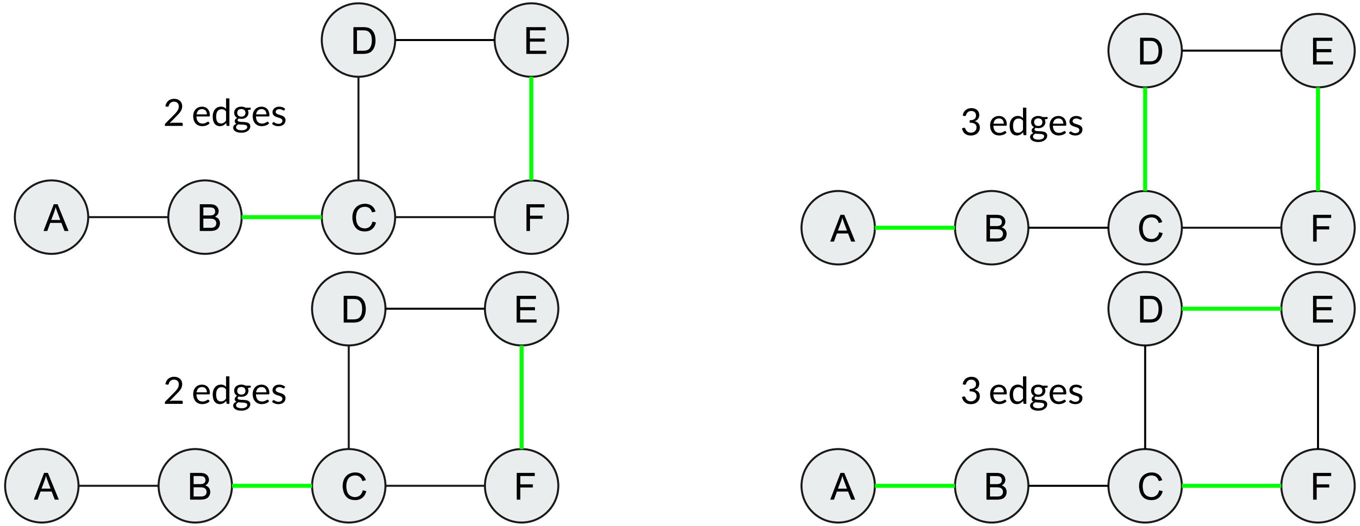






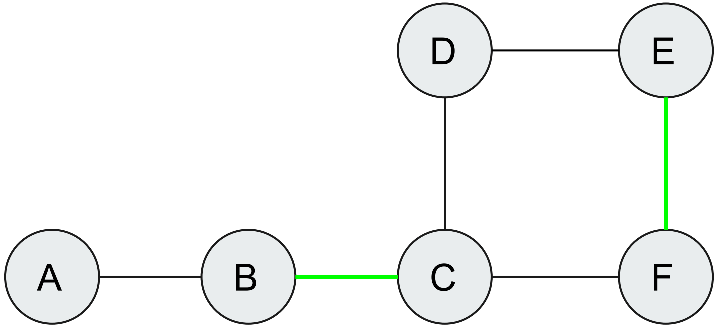
Maximum Matching

- A **maximum matching** is a matching with the greatest number of edges.
- All maximum matchings are maximal, but not all maximal matchings are maximum



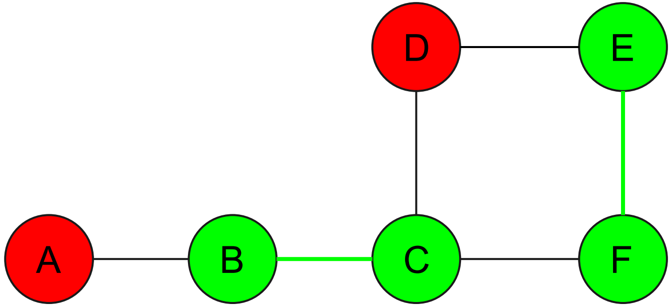
Alternating and Augmenting Paths

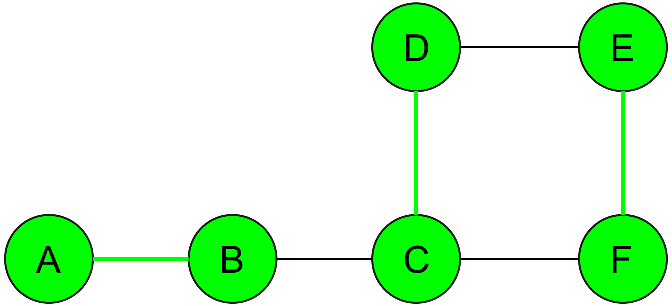
- A *M*-**alternating path** is a path that alternates between edges in *M* and not in *M*
- Note: all vertices except start and end must be saturated
- E.g:
  - AB,BC,CF,FE, ED
  - BC,CF
  - DE,EF,FC



Augmenting Paths

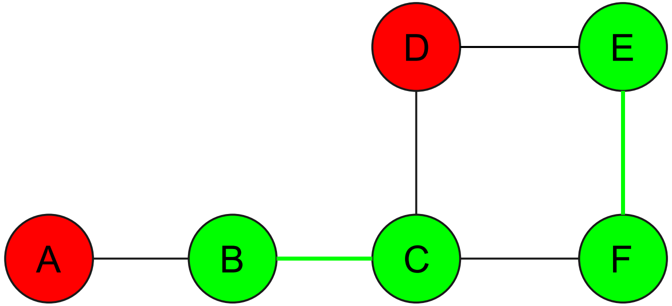
- A *M*-**augmenting path** is a *M*-alternating path that starts and ends in an unsaturated vertex.

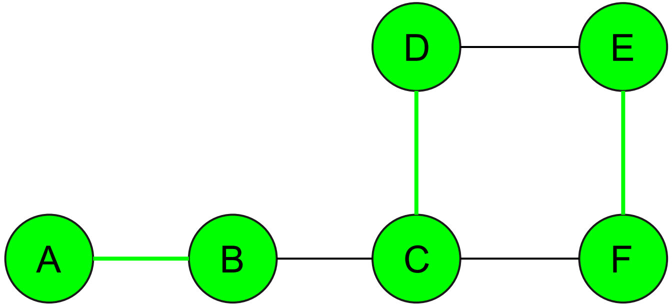




Using Augmenting Paths

- Finding *M*-augmented paths can create better matchings
- Find *M*-augmented path from *x* to *y*, flip state of all edges in *M* from *x* to *y*
- E.g: A->B->C->D



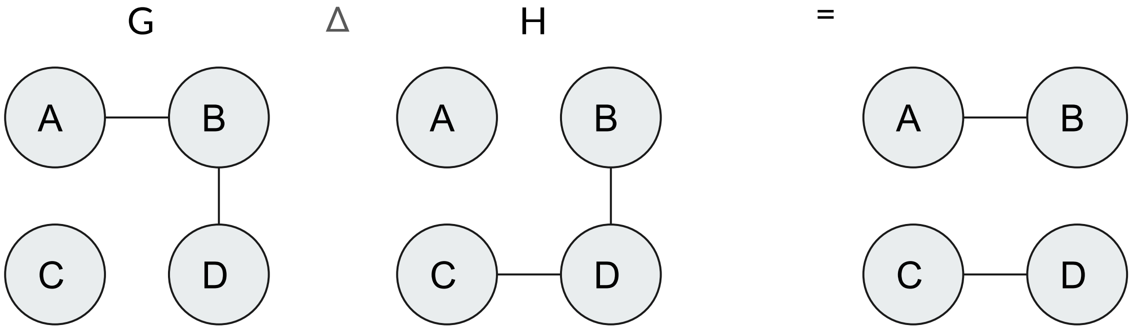


Berge's Theorem

Berge's Theorem: A matching M is a maximum matching iff G has no M-augmenting path.

Symmetric Difference

- **Symmetric difference** between *G* and *H*, or G Δ H, is the subgraph of *G* ∪ *H* whose edges are the edges that appear in only one of *G* and *H*.
- **xor** equivalent for graphs
- Formally, how augmenting paths are augmented



Berge’s Theorem Proof

Lemma 1: Consider 2 matchings *M* and *M’*.Let *G’* be $M \Delta M'$. All connected components of *G’* must either be isolated vertices, even cycles alternating between *M* and *M’*, or paths alternating between *M* and *M’*

Proof: Each vertex in *G’* can have at max degree 2. Therefore, *G’* can only be made of paths and cycles. Also, all cycles in *G’* must alternate between *M* and *M’*, so they must all be of even length.

Berge's Theorem Proof

Berge's Theorem: A matching *M* is a maximum matching iff *G* has no *M*-augmenting path.

Proof by contraposition: *G* has a *M*-augmenting path iff *M* is not a maximum matching.

Forwards Proof: Follows by augmenting using path *P*. Let *M'* be P Δ *M*, *M'* is a larger matching.

Backwards Proof: Let *M'* be a larger matching than *M*. From Lemma 1, *M* Δ *M'* contains only paths and even cycles. Since *M'* has more edges than *M*, one of the components must have more edges from *M'* than *M*, so it must be a path. Since it alternates, and starts and ends with unsaturated vertices in *M*, it must be a *M*-augmenting path.

Maximum Matching Algorithm

- Find all augmenting paths
- This version only applicable to bipartite graphs

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procedure MATCHBIPARTITE(X, Y-bigraph G)

    $M \leftarrow \emptyset$ ▷ M initially empty

    do

        $P \leftarrow \text{AugPathAlg}(G, M)$ ▷ New augmented path found with M, G

        $M \leftarrow M \Delta P$ ▷ Symmetric difference between M, P

    while $P \neq \emptyset$

    return M

Finding Augmenting Paths

procedure AugPathAlg(X, Y-bigraph G and matching $M = (V_M, E_M)$)

$G' \leftarrow G$

Orient $G' : \forall e \in E_M : e(x_i, y_j) = e(y_j \rightarrow x_i); \forall e \notin E_M : e(x_i, y_j) = e(x_i \rightarrow y_j)$

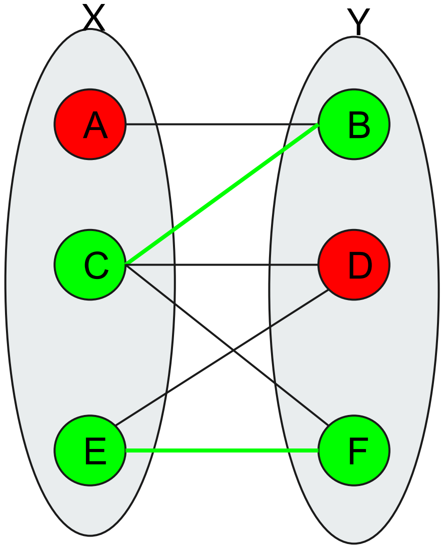
Add vertex s to G' with edges $\forall x_i \in X, x_i \notin V_M : (s \rightarrow x_i)$

Add vertex t to G' with edges $\forall y_j \in Y, y_j \notin V_M : (y_j \rightarrow t)$

$P \leftarrow$ ShortestPathBFS(G', s, t) ▷ Use BFS to find shortest path from s to t

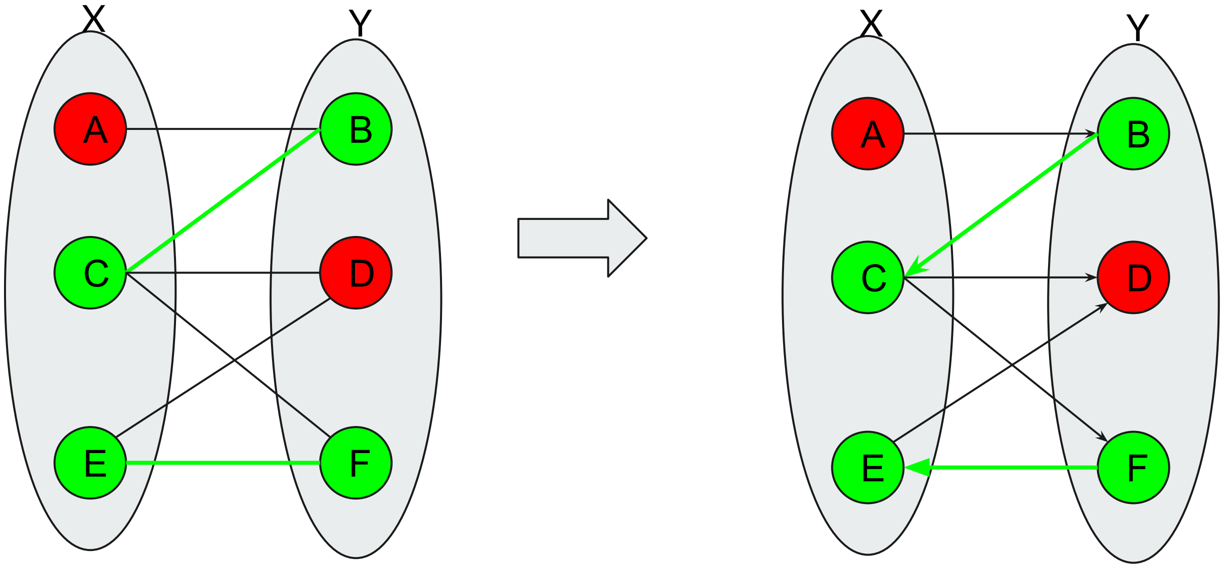
return $P - \{e(s, x_i), e(y_j, t)\}$ ▷ Return path without added edges

Finding Augmenting Paths: Example



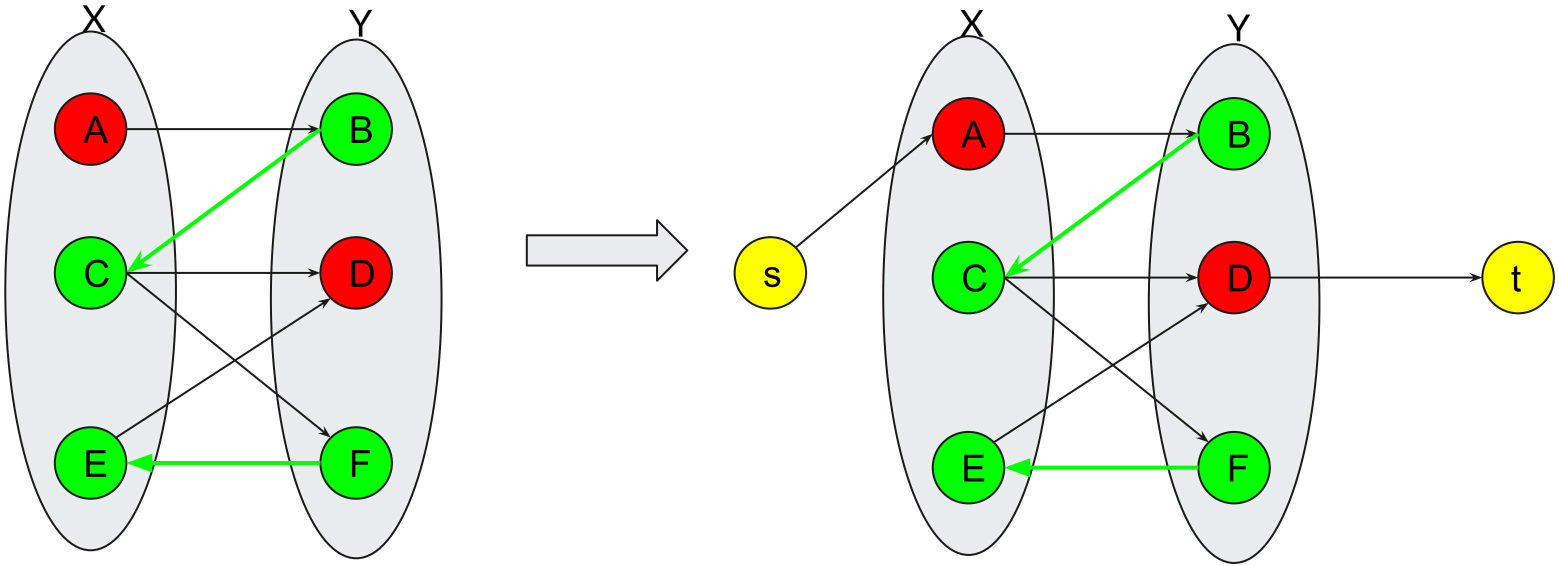
Finding Augmenting Paths: Example

1: Orient the Graph: Edges in *M* going backwards, edges not in *M* going forwards.



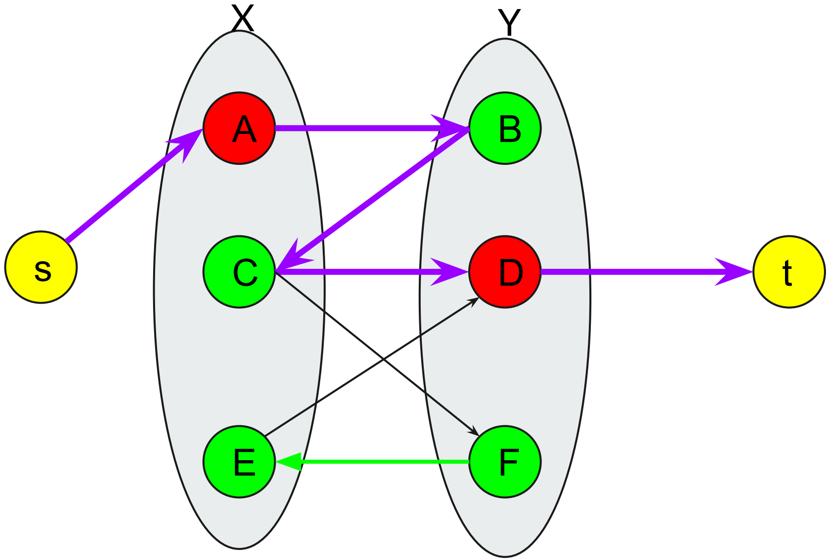
Finding Augmenting Paths: Example

2: Add a source *s*, going to all unsaturated vertices in *X*, and sink *t*, with a connection coming from all unsaturated vertices in *Y*



Finding Augmenting Paths: Example

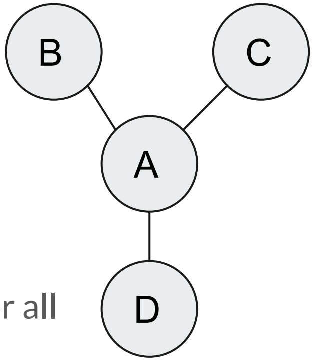
3: Run BFS from *s* to *t*, return path without *s* and *t*



Path found:
s->A->B->C->D->t

Return augmenting path A->B->C->D

Is a perfect matching possible?



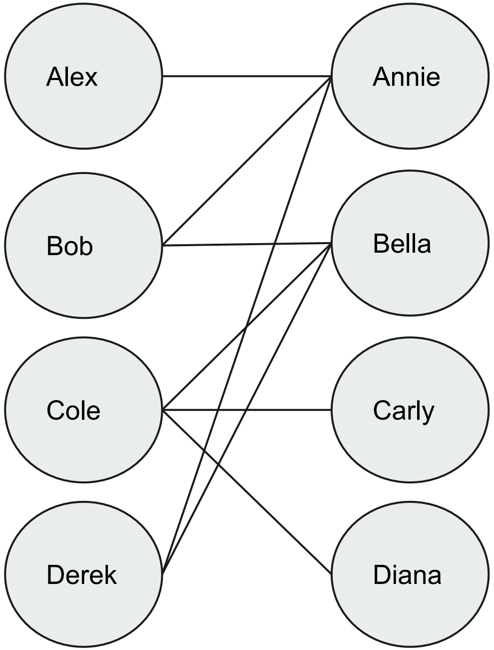
Hall's Theorem: An X,Y bipartite graph has a matching that saturates X iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

To show that a graph G does not have a matching that saturates X, we need to find a subset $S \subseteq X$ where $|N(S)| < |S|$

Motivating Example: A Marriage Problem

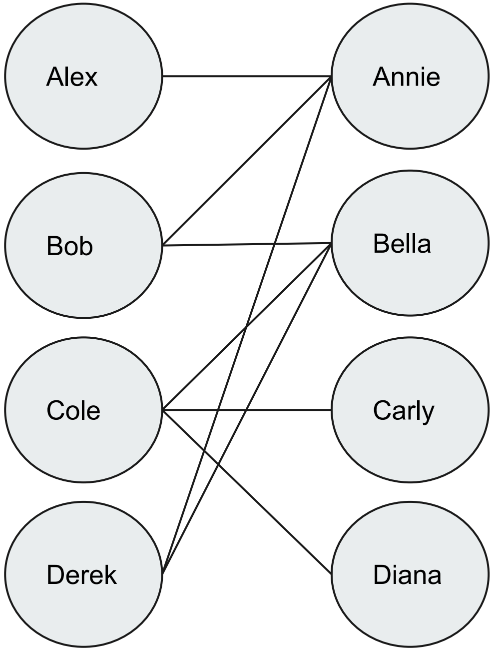
- Suppose there are two groups, X with n men and Y with n women. There exists an edge between man x and woman y if x and y would happily marry. Does there exist a pairing between men and women such that everyone is happily married?

Example



Does there exist a perfect matching?

Example



Consider subset S= {Alex, Bob, Cole}

Marriage partners for this subset are only {Annie,Bella}

Since $|N(S)| < |S|$, no perfect matching exists.

Hall's Theorem Proof

Hall's Theorem: An X,Y bipartite graph has a matching that saturates X iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

Forward direction Proof: Let M be a matching that saturates X. For any subset S of X, let $M(S)$ be the vertices in Y matched to S. By definition, $|M(S)| = |S|$. Additionally, since $M(S) \subseteq N(S)$, $|N(S)| \geq |M(S)|$. Therefore, $|N(S)| > |S|$.

Hall’s Theorem Proof

Hall’s Theorem: An X,Y bipartite graph has a matching that saturates X iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

Backward direction Proof: Let M be a maximum matching (which does not saturate X). Let u be an unsaturated vertex in X. Consider all M-alternating paths starting from u. Let Z be vertices in Y reachable from these paths, and let W be vertices in X reachable from these paths. Every vertex z in Z must be paired to a vertex in W - u, because if it wasn’t, the path from u to z would be an augmenting path. Every vertex in W must also be paired to a vertex to Z, because u must have reached it through an edge in M. Therefore, there exists a mapping between W-u and Z, demonstrating that $|W| = |Z|+1$. In addition, $N(W) \subseteq Z$. To demonstrate this, we select any neighbor in Y of any w in W, which we call y. If edge (w,y) is in M, y is in Z by the pairing previously shown. If edge (w,y) is not in Z, there exists an augmenting path from u to w to y, which is a contradiction. This shows that $N(W) \subseteq Z$. Since $|N(W)| \leq |Z| = |W|-1$, $|N(W)| \leq |W|$.