9.1 Matching in General Graphs

For the most part, we've discussed matching restricted to bipartite graphs. We're going to generalize it now to all graphs. First define the function $o(G)$ as the number of **odd connected components** in G. As we know, odd connected components have an odd number of vertices. In order for a graph to have a perfect matching, we'll use what could loosely be considered as a generalization of Hall's Condition.

Tutte's Theorem states that a graph G with a perfect match satisfies the inequality $\forall S \subseteq V(G) : o(G - S) \leq |S|$. Formally, a graph $G = (V, E)$ has a perfect matching if and only if for every possible vertex set $S \subseteq V(G)$, the subgraph induced by $V - S$ has at most $|S|$ connected components with an odd number of vertices. Let's develop a proof for Tutte's Theorem.

9.2 Independent Sets and Covers

A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint on all $e \in E(G)$. The vertices in Q *cover* $E(G)$. An **edge cover** of G is a set $L \subseteq E(G)$ such that L has at least one edge incident on all $v \in V(G)$. The edges in L *cover* $V(G)$.

The **König-Egerváry Theorem** states that if G is a bipartite graph, then the size of a maximum matching in G equals the minimum size of a vertex cover.

An **independent set** of vertices on a graph G are a set of vertices that are not connected by an edge. The size of a maximum independent set on G is called the **independence number** of G. For a bipartite graph, this isn't necessarily the size of the larger partite set.

In G, $S \subseteq V(G)$ is an independent set if and only if $\bar{S}$ is a vertex cover. Thus a maximum independent set is the complement of a minimum vertex cover, and their sizes summed equals the order of G.

9.3 Vertex Connectivity

So far we've talked about connectivity for undirected graphs and weak and strong connectivity for directed graphs. For undirected graphs, we're going to now somewhat generalize the concept of connectedness in terms of network robustness. Essentially, given a graph, we may want to answer the question of how many vertices or edges must be removed in order to disconnect the graph; i.e., break it up into multiple components.

Formally, for a connected graph G, a set of vertices $S \subseteq V(G)$ is a **separating set** if subgraph $G - S$ has more than one component or is only a single vertex. The set S is also called a **vertex separator** or a **vertex cut**. The **connectivity** of G, $\kappa(G)$, is the minimum size of any $S \subseteq V(G)$ such that $G - S$ is disconnected or has a single vertex; such an S would be called a **minimum separator**. We say that G is k-**connected** if $\kappa(G) \geq k$.

Consider a **hypercube** Q_k, which is the simple graph whose vertices can be uniquely labeled by the k-tuple with entries in $\{0, 1\}$ and whose edges go between vertices with labels that differ by at most 1 entry. We can use induction to prove that the hypercube Q_k is k-connected.

9.4 Edge Connectivity

We have similar concepts for edges. For a connected graph G, a set of edges $F \subseteq E(G)$ is a **disconnecting set** if $G - F$ has more than one component. If $G - F$ has two components, F is also called an **edge cut**. The **edge-connectivity** if G, $\kappa'(G)$, is the minimum size of any $F \subseteq E(G)$ such that $G - F$ is disconnected; such an F would be called a **minimum cut**. A **bond** is a *minimal* non-empty edge cut; note that a bond is not necessarily a minimum cut. We say that G is k-**edge-connected** if $\kappa'(G) \geq k$. In a couple classes, we'll talk about how one might find a minimum cut in an arbitrary graph.

For a simple graph, we can show that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

We can also show that an edge cut F is a bond iff $G - F$ has exactly two components.