10.1 2-Connected Graphs

We're going to talk more specifically about 2-connected and 2-edge-connected graphs. We can characterize them using **internally disjoint** paths. Two u, v-paths are internally disjoint if there is no common internal vertex. Similarly, two u, v-paths are internally edge-disjoint if there is no common internal edge. **Whitney** proved that a graph G of at least three vertices is 2-connected if and only if for all $u, v \in V(G)$ there exists at least two internally disjoint u, v-paths. We'll also prove this.

Additionally and equivalently:

- G is connected and has no cut vertex
- $\forall u, v \in V(G)$ there exists some cycle $C \in G : u, v \in C$
- $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle

A **subdivision** of an edge (u, v) is the operation of replacing (u, v) with two edges attached to a new vertex, i.e., (u, w) and (v, w). Subdividing any arbitrary edge in a 2-connected graph will not affect the graph's 2-connectivity.

An **ear decomposition** of G is a decomposition of the edges of G into a sequence of paths $P_0, P_1, \ldots, P_k$, where P_0 is a closed path (cycle) and for $i \geq 1$ P_i has unique endpoints in $P_0 \cup \ldots \cup P_{i-1}$. These P are called **ears** or **open ears**. A graph is 2-connected if and only if it has an ear decomposition and every cycle in a 2-connected graph is the initial cycle in some ear decomposition. We'll use the idea of subdivisions in our proof of the preceding sentence.

A **closed-ear decomposition** of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is a path with unique or non-unique endpoints in $P_0 \cup \ldots \cup P_{i-1}$. These P are called **closed ears**. A graph is 2-edge-connected if and only if it has a closed-ear decomposition and every cycle in a 2-edge-connected graph is the initial cycle in some closed ear decomposition.

Note that every 2-connected graph is necessarily 2-edge-connected.

10.2 Biconnectivity

A graph that has no cut vertices is also called **biconnected**. We differentiate between 2-connected here in that the graphs K_1 and K_2 would also be considered biconnected even if they aren't 2-connected. The **biconnected components** (BiCCs) of a connected (but not necessarily biconnected) graph are the maximal subgraphs of the graph that are themselves biconnected. These are also called **blocks**. A vertex that connects to different blocks is called an **articulation point** or a **cut-vertex**. A **block-cutpoint graph** is a bipartite graph where one partite set consists of cut-vertices and one partite set consists of contracted representations of of every BiCC.

10.3 Digraph Connectivity

We can extend the concepts and terminology of connectivity to directed graphs as well. A **vertex cut** or **separating set** in a digraph D is a set $S \subseteq V(G)$ such that $D - S$ is not strongly connected. The **connectivity** $\kappa(D)$ is the minimum size of vertex set S such that $D - S$ is not strongly connected or is a single vertex. If $k \leq \kappa(D)$, then D is k-**connected**. A digraph is k-edge-connected if every **edge cut** has at least k edges, where an edge cut separates $V(D)$ into two sets $S, \bar{S}$ such that the size of the edge cut is the number of directed edges (u, v) from $v \in S$ to $u \in \bar{S}$. The **edge-connectivity** $\kappa'(D)$ is the minimum size of an edge cut. If $k \leq \kappa'(D)$, then D is k-**edge-connected**.

As we have noted, 2-edge-connected graphs share similarities with strongly connected digraphs. We can show that adding a directed ear to a strong digraph produces a larger strongly connected digraph. We can also show that a graph has a **strong orientation** if and only if it is 2-edge-connected.