

12.1 Network Flow

Consider a directed graph G where each edge $e \in E(G)$ has a given **capacity** $c(e)$. We also have a distinguished **source vertex** s and **sink vertex** t . Such a graph is called a **flow network**.

A **flow** $f(e)$ on a flow network G assigns a value to each $e \in E(G)$. For each $v \in V(G)$ we have $f^+(v)$ as the sum of flows from incoming edges on v and $f^-(v)$ as the sum of flows on outgoing edges. For non-source and non-sink vertices, a flow is **feasible** if it satisfies constraints $\forall e \in E(G) : 0 \leq f(e) \leq c(e)$ and $\forall v \in V(G), v \neq s, t : f^+(v) = f^-(v)$. The **value** $\text{val}(f)$ of a flow f is the net flow into the sink, $f^-(t) - f^+(t)$. A **maximum flow** is a feasible flow where $\text{val}(f)$ is maximum.

When f is a feasible flow in a network, a **f -augmenting path** is a source-to-sink path P where for each $e \in P$:

1. if P follows e in a forward direction, then $f(e) < c(e)$
2. if P follows e in a backward direction, then $f(e) > 0$

Define $\epsilon(e) = c(e) - f(e)$ when e is forward on P and $\epsilon(e) = f(e)$ when e is backward on P . The **tolerance** of P is $\min_{e \in E(P)} \epsilon(e)$.

If P is an f -augmenting path with tolerance z , then changing flow by $+z$ on forward edges in P and $-z$ on backward edges in P produces a new feasible flow $\text{val}(f') = \text{val}(f) + z$.

In a flow network, a **source-sink cut** $[S, T]$ consists of the edges between a **source set** S and **sink set** T , where S and T partition the nodes and $s \in S, t \in T$. The **capacity** of the cut $[S, T]$, $\text{cap}(S, T)$ is the total capacities of the edges of $[S, T]$, with the net flow from S to T equal to $\text{val}(f)$ and $\text{val}(f) \leq \text{cap}(S, T)$. Among all possible $[S, T]$ cuts, the one with the lowest $\text{cap}(S, T)$ gives us a bound on our maximum flow. The **Max-flow Min-cut Theorem** states the duality between the maximum flow and **minimum cut** problems; specifically, the maximum value of a feasible flow equals the minimum capacity of a source-sink cut.

12.2 Max Flow – Edmonds-Karp Algorithm

At a high level, the iterative algorithm for identifying f -augmenting paths to incrementally increase the flow in a network is called the **Ford-Fulkerson Algorithm**. When we explicitly use BFS to find the shortest of such paths, we have the **Edmonds-Karp Algorithm**. We define this algorithm above for max flow. Should we wish to find a min cut instead, we can use the set of vertices visited by our BFS before termination as our S

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procedure EDMONDS-KARP(Flow Network  $G(V, E^+, E^-, C, s, t)$ )
     $\triangleright C =$  edge capacities,  $s =$  source vertex,  $t =$  sink vertex
    for all  $e \in E(G)$  do
         $F(e) \leftarrow 0$   $\triangleright$  Initialize flows to zero
    do  $\triangleright$  Do iterative BFS searches for  $f$ -augmenting paths
        for all  $v \in V(G)$  do
             $parent(v) \leftarrow -1$ 
         $Q \leftarrow s, Q_n \leftarrow \emptyset$ 
        while  $Q \neq \emptyset$  do
            for all  $v \in Q$  do
                for all  $u \in N^+(v) \cup N^-(v) : parent(u) = -1$  do
                     $e \leftarrow (v, u)$ 
                    if  $(F(e) < C(e) \text{ and } u \in N^+(v))$  or  $(F(e) > 0 \text{ and } u \in N^-(v))$  then
                         $parent(u) = v, Q_n \leftarrow u$ 
                swap( $Q, Q_n$ ),  $Q_n \leftarrow \emptyset$ 
            if  $parent(t) = -1$  then  $\triangleright$  Did we find path to sink?
                 $foundpath \leftarrow \text{false}$ 
            else
                 $foundpath \leftarrow \text{true}, tol \leftarrow \infty, v \leftarrow t$ 
                while  $v \neq s$  do  $\triangleright$  First determine tolerance  $tol$ 
                     $u \leftarrow parent(v), e \leftarrow (u, v)$ 
                    if  $e \in E^+(G)$  then
                         $tol \leftarrow \min(tol, C(e) - F(e))$ 
                    else
                         $tol \leftarrow \min(tol, F(e))$ 
                 $v \leftarrow t$ 
                while  $v \neq s$  do  $\triangleright$  Now use tolerance to update flows
                     $u \leftarrow parent(v), e \leftarrow (u, v)$ 
                    if  $e \in E^+(G)$  then
                         $F(e) \leftarrow F(e) + tol$ 
                    else
                         $F(e) \leftarrow F(e) - tol$ 
            while  $foundPath = \text{true}$ 
        return  $(F^-(t) - F^+(t))$ 

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source set, unvisited vertices as the T sink set, and therefore the edges cut between them $[S, T]$ is our minimum cut.