20.1 Line Graphs

The **line graph** of G, written as $L(G)$, is the simple graph whose vertices are the edges of G. Edges $v, u \in E(G)$, represented as vertices $v, u \in V(L(G))$, have an edge between them in $L(G)$ if they share a common endpoint $w \in V(G) : \{(v, w), (u, w)\} \in E(G)$. There is a relationship between problems involving edges in G and problems involving the vertices in $L(G)$:

1. An Eulerian Circuit in G is a spanning cycle in $L(G)$.
2. A matching in G is an independent set in $L(G)$.
3. A cut edge $e = (u, v)$ in G is a cut vertex in $L(G)$ if $d(u), d(v) > 1$.
4. Edge-coloring in G is equivalent to vertex coloring in $L(G)$.

Consider some H such that $L(H) = G$, and, while we aren't going to discuss it today, we can find such an H for G in polynomial time if it exists. An interesting notion is that we can theoretically exploit this relationship. Note that it's possible to compute a maximum matching in polynomial time in general on H while a maximum independent set requires exponential time in general on G; however there is a 1-to-1 correspondence between the results. Likewise, the same might be said of finding an optimal vertex coloring in general takes exponential time while an optimal edge coloring only requires polynomial time. We going to focus on the last problem today. Edge-coloring in G and how it relates to vertex coloring in $L(G)$.

20.2 Edge-coloring

Edge coloring is the problem of assigning labels, i.e. *colors*, to all $e \in E(G)$ such that no two edges have the same color if they share an endpoint $v \in V(G)$. We use similar terminology as with vertex coloring. A coloring is **proper** if it satisfies the above criteria. We consider a k-**edge-coloring** to be a proper edge coloring of k colors. The **edge-chromatic-number** or **chromatic index**, $\chi'(G) = k$, is equal to the smallest k for which G is properly k-edge-colorable. Let's consider some bounds on $\chi'(G)$.

Since all edges incident on the largest degree vertex require separate colors, obviously $\chi'(G) \geq \Delta(G)$.

If we consider a greedy scheme to color edges and note that no edge shares endpoints with more than $2\Delta(G) - 1$ edges, we have the bound $\chi'(G) \leq 2\Delta(G) - 1$.

As a greedy edge coloring scheme on G is equivalent to a greedy vertex coloring scheme on $L(G)$, we further have the bounds $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$.

If G is bipartite, we can show that $\chi'(G) = \Delta(G)$.

For any simple graph, we can further show that $\chi'(G) \leq \Delta(G) + 1$. Or combined with our lower bound, $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.