3.1 Decomposition and Special Graphs

The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set. Two vertices are adjacent in $\overline{G}$ iff they are not adjacent in G. A graph G is called **self-complementary** if G is isomorphic to $\overline{G}$. A **decomposition** of a graph is a list of subgraphs such that each edge appears in only a single subgraph in the list.

There are a couple specific names for certain graphs that we might use (and have already used) repeatedly:
Triangle: A three vertex cycle C_3 or clique K_3
Claw: The complete bipartite graph $K_{1,3}$
Note: the claw is also a **star graph**, which are the class of complete bipartite graphs $K_{1,n}$
Also note: the book gives several other examples in 1.1.35; we probably won't be talking much specifically about the other ones.

3.2 Walks and Connectivity

A **walk** is a list of of vertices and edges (e.g., v_0, e_5, v_6, e_1, v_2) such that each listed edge connects the preceding and proceeding listed vertices. The list begins and ends with vertices. A **trail** is a walk with no repeated edges. A **path** has no repeated edges or vertices. A u, v-walk and u, v-trail begin with vertex u and end with vertex v. A u, v-path is a path with endpoint vertices u and v having degree 1 and all other vertices being internal. The **length** of a walk/trail/path is the number of contained edges. A walk is **closed** if the start and end vertices are the same. *Random walks* performed by starting at a given vertex, moving to an adjacent vertex selected at random, then iteratively continuing this procedure from the newly selected vertex have a number of interesting uses and properties; we'll talk a little more about these later.

A graph G is **connected** if for every $u, v \in V(G)$ there is a path connecting u and v. Otherwise G is disconnected. A **connected component** of G is a *maximal* connected subgraph. We say a component is **trivial** when it consists of a single vertex and no edges. Otherwise, the component is **nontrivial**. A **cut-edge** or **cut-vertex** are the edges or vertices that, when removed from G, increase the number of connected components.

We'll talk more about connectivity later in the course.

3.3 Induction

In this class, we'll often be proving properties about graphs using **induction** and **necessity and sufficiency**.

Review: **Weak Induction** as a proof method. You have probably seen this in an earlier class. Consider a natural number n, let $P(n)$ be a mathematical statement. If properties 1 and 2 below hold, then $P(n)$ is true for all $n \in \mathbb{N}$.

1. $P(1)$ is true
2. for $k \in \mathbb{N}$, if $P(k)$ is true, then $P(n = k + 1)$ is true

1 is the **basis step** and 2 is the **inductive step**. The inductive step includes our **induction hypothesis**, which is the assumption that our current step of $P(k)$ is true. The basis step might utilize $P(0)$ or multiple k for $P(k)$.

Weak inductive proofs then work to show that if e.g. $P(1)$ is true, and $P(k) \implies P(k+1)$ is true, then $P(1) \implies P(1+1)$, $P(1+1) \implies P(1+1+1)$, etc. So $P(k)$ is true for all natural numbers.

Prove that: $2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$

Basis Step: $P(n = 1) = 2^1 = 2^2 - 2 = 2$ ✓
Induction Step: $P(n = k + 1) = 2^1 + 2^2 + \ldots + 2^k + 2^{k+1}$
$= [2^1 + 2^2 + \ldots + 2^k] + 2^{k+1}$
$= 2^{k+1} - 2 + 2^{k+1}$
$= [2^{k+1} + 2^{k+1}] - 2$
$= 2 \times 2^{k+1} - 2$
$= 2^{k+2} - 2$
$= 2^{(k+1)+1} - 2$
$= 2^{n+1} - 2$ ✓

However, a number of graph proofs require **Strong Induction**:

1. $P(1)$ is true
2. for $k > 1$, if $P(k)$ is true, then $P(n)$ is true for $1 \leq k < n$

Instead of limiting ourselves to $P(n = k + 1)$ in our inductive step, we assume that all $P(k)$ less than our n are true. When we're working with graphs, we often do induction on the number of vertices/edges in some graph G. With strong induction, we can establish a relation between $G(k)$ and $G(n)$ as the difference of a larger subgraph beyond just a single vertex or a single edge. This is often required, as a key aspect of induction is showing how $P(k)$ being true implies $P(n)$ is true.

By using only weak induction, we'd need to consider all possible "structural" ways that adding a single vertex/edge to get from $P(k)$ to $P(n)$ might impact the property we're trying to prove. Because of the *combinatorial explosion*[1] of possible graph configurations, this becomes quite unwieldy quite quickly. With strong induction, we can often specifically select a vertex or edge to remove from $P(n)$ to get to $P(k)$; we then only need to consider the impact of that specific structure within our proof. This might be confusing, but we'll be doing a number of proofs throughout the rest of the class to differentiate and emphasize this key difference.

Necessity and sufficiency is used to prove *equivalence relationships*, such as we'll soon show that *a graph is bipartite iff it has no odd cycle* (note: iff $\rightarrow$ if and only if). To generally prove an equivalence relationship, we can show that the given conditions (A iff B) are both necessary and sufficient; i.e., by proving that if condition A implies condition B and if conditions B implies condition A, we prove their equivalence. For any equivalence relationship, knowledge about one condition gives us knowledge about the other condition – so if we know a graph has no odd cycles, we also know that it therefore must be bipartite.

3.4 More on Walks and Cycles

An **even** walk/path/trail/cycle has an even length, or number of edges. Likewise, an **odd** walk/path/trail/cycle has an odd length, or number of edges. An **even graph** has all vertex degrees even. A vertex is even if it has an even degree or odd when it has an odd degree.

Prove with induction: Every closed odd walk contains an odd cycle.

Basis Step: $P(l = 1)$: a length 1 walk is a single loop, hence is an odd cycle of length 1
Induction Step: $P(l = n > 1)$: We use the induction hypothesis to assume walks of length $k < n$ have an odd cycle. Consider walk W. If W is odd and has no repeated vertices, then W is an odd cycle by itself. Otherwise, some vertex v is repeated. Consider breaking W into walks W_1 and W_2 originating from v. As the length of W is odd, then W_1 must be odd and W_2 even. We say that the length of W_1 is $k < n$, which by our inductive hypothesis must have some odd cycle. As W_1 is a subpath of W, then W must also contain an odd cycle. This is the power of strong induction in action!!!

Prove with necessity and sufficiency: A graph is bipartite iff it has no odd cycle.

[1]