26.1 Face Coloring and Hamiltonian Cycles

A **face coloring** is the coloring of each face of a planar graph. A face coloring is **proper** if each pair of faces that share an edge have differing colors. This is equivalent to the map coloring problem we discussed with regards to planar graph vertex coloring and the 4/5-color theorems. As such, we can consider face coloring of a planar graph G as equivalent to vertex coloring of its dual graph G^*.

Similarly, like we used triangulations in our 4/5-color theorems, if we can show that all dual graphs of triangulations are 4-face-colorable, we equivalently show that all triangulations are 4-colorable. As all planar graphs are subgraphs of some triangulation, this would effectively give us a proof of the 4-color theorem.

Tait's Theorem states that a simple 2-edge-connected 3-regular plane graph is 3-edge-colorable if and only if it is 4-face-colorable. Such a 3-edge-coloring of a 3-regular graph is referred to as a **Tait coloring**. Showing that every 2-edge-connected 3-regular planar graph is 3-edge-colorable can be reduced to showing every 3-connected 3-regular graph is 3-edge-colorable. Thus, the 4-color Theorem reduces to finding Tait colorings of 3-edge-connected 3-regular planar graphs. The statement of the existence of such colorings was referred to as **Tait's Conjecture**.

We can now consider the similarities between the existence of a Hamiltonian cycle in a graph and its colorability. First, note that every Hamiltonian 3-regular graph has a Tait coloring. While it was first assumed by Tait that every 3-connected 3-regular planar graph had a Hamiltonian cycle (thus proving the 4-color theorem), this is not the case. **Grinberg's Theorem** gives a necessary condition for the existence of a Hamiltonian cycle in such graphs:

If G is a loopless plane graph with Hamiltonian cycle C, and G has $f_i^{'}$ faces of length i inside C and $f_i^{''}$ faces of length outside C, then $\sum_i (i-2)(f_i^{'} - f_i^{''}) = 0$.

This gives us another necessary (but not sufficient) condition for a class of Hamiltonian graphs. Consider applying this condition to the below graph, which is the smallest known non-Hamiltonian 3-connected 3-regular planar graph.