6.1 Directed Acyclic Graphs

A graph with no cycle is acyclic. Directed acyclic graphs, or DAGs are acyclic directed graphs where vertices can be ordered in such a way that no vertex has an edge that points to a vertex earlier in the order. This also implies that we can permute the adjacency matrix in a way that it has only zeros on and below the diagonal. This is a strictly upper triangular matrix. Arranging vertices in such a way is referred to in computer science as topological sorting.

6.2 Trees

A forest is an acyclic graph. A tree is a connected undirected acyclic graph. If the underlying graph of a DAG is a tree, then the graph is a polytree. A leaf is a vertex of degree one. Every tree with at least two vertices has at least two leaves. A spanning subgraph of $G$ is a subgraph that contains all vertices. A spanning tree is a spanning subgraph that is a tree. All graphs contain at least one spanning tree.

Necessary properties of a tree $T$:

1. $T$ is (minimally) connected
2. $T$ is (maximally) acyclic
3. $T$ has $n - 1$ edges
4. $T$ has a single $u, v$-path $\forall u, v \in V(T)$
5. Any edge in $T$ is a cut edge; any non-leaf vertex is a cut vertex
6. Adding any edge to a tree (without adding vertices) creates a cycle

Which of these properties are also sufficient?

Generally, because any possible tree can be constructed by iteratively adding vertices and edges to the simplest possible tree (single vertex/single edge), we can often use weak (instead of strong) induction in proofs on trees.

Prove using weak induction that every tree is bipartite.

Prove using strong induction on the number edges $m$ of a tree $T$: Between any two $u, v \in V(T)$, there exists only a single unique $u, v$-path. Note that the approach in this proof is nearly identical to what we would do with weak induction.
Note: In prior classes (especially the CS students), you might have used structural induction in the context of proofs on trees. The difference with structural induction is that there does not need to be an explicit “countable” variable on which induction is performed, though the recursive idea behind the proof technique is similar to strong induction. We won’t be covering structural induction in this course.

6.3 Distances

The distance between a $u$ and $v$ in $G$ written as $d(u,v)$ is the length of the shortest $u,v$-path. Remember that the diameter of a graph is $\max_{u,v \in V(G)}d(u,v)$, or the maximum $d(u,v)$ among all possible $u,v$ pairs. The eccentricity of a vertex $\epsilon(u)$ is $\max_{v \in V(G)}d(u,v)$, or maximum $u,v$ path from that $u$. The radius of a graph is the minimum eccentricity of any vertex, or $\min_{v \in V(G)}d(u,v)$. The center of a graph is the subgraph induced by the vertices of minimum eccentricity.