13.1 2-Connected Graphs

We’re going to talk more specifically about 2-connected and 2-edge-connected graphs. We can characterize them using internally disjoint paths. Two \( u, v \)-paths are internally disjoint if there is no common internal vertex. Similarly, two \( u, v \)-paths are internally edge-disjoint if there is no common internal edge. Whitney proved that a graph \( G \) of at least three vertices is 2-connected if and only if for all \( u, v \in V(G) \) there exists at least two internally disjoint \( u, v \)-paths. We’ll also prove this. Additionally and equivalently:

- \( G \) is connected and has no cut vertex
- \( \forall u, v \in V(G) \) there exists some cycle \( C \in G : u, v \in C \)
- \( \delta(G) \geq 1 \) and every pair of edges in \( G \) lies on a common cycle

A subdivision of an edge \((u, v)\) is the operation of replacing \((u, v)\) with two edges attached to a new vertex, i.e., \((u, w)\) and \((v, w)\). Subdividing any arbitrary edge in a 2-connected graph will not affect the graph’s 2-connectivity.

An ear decomposition of \( G \) is a decomposition of the edges of \( G \) into a sequence of paths \( P_0, P_1, \ldots, P_k \), where \( P_0 \) is a closed path (cycle) and for \( i \geq 1 \) \( P_i \) has unique endpoints in \( P_0 \cup \ldots \cup P_{i-1} \). These \( P \) are called ears or open ears. A graph is 2-connected if and only if it has an ear decomposition and every cycle in a 2-connected graph is the initial cycle in some ear decomposition. We’ll use the idea of subdivisions in our proof of the preceding sentence.

A closed-ear decomposition of \( G \) is a decomposition \( P_0, \ldots, P_k \) such that \( P_0 \) is a cycle and \( P_i \) for \( i \geq 1 \) is a path with unique or non-unique endpoints in \( P_0 \cup \ldots \cup P_{i-1} \). These \( P \) are called closed ears. A graph is 2-edge-connected if and only if it has a closed-ear decomposition and every cycle in a 2-edge-connected graph is the initial cycle in some closed ear decomposition.

Note that every 2-connected graph is necessarily 2-edge-connected.

13.2 Biconnectivity

A graph that has no cut vertices is also called biconnected. We note that graphs \( K_1 \) and \( K_2 \) would also be considered biconnected even if they aren’t 2-connected by our prior characterizations. The biconnected components (BiCCs) of a connected (but not necessarily biconnected) graph are the maximal subgraphs of the graph that are themselves biconnected. These are also called blocks. A vertex that connects to different blocks is called an articulation point or simply a cut vertex. A block-cutpoint graph is a bipartite graph where one partite set consists of cut-vertices and one partite set consists of contracted representations of of every BiCC. Edges in this bipartite graph represent which articulation points connect which blocks.