Lecture 19 - Extremal Coloring

Thursday, March 31, 2022 9:54 AM

Slota's mood: chicken katsu curry

Exam average: 76

Target average: 80

curving: TBD

Overall average: ~80

Properties of a k-chromatic graph

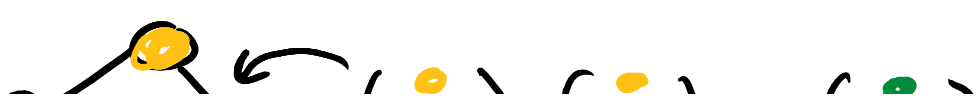
How small can a k-chromatic graph be?

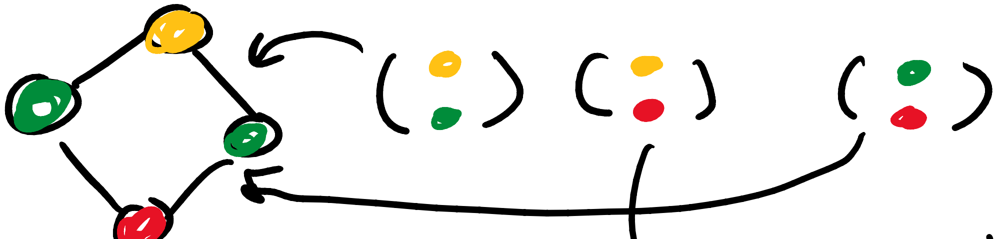
- Consider all possible color pairs:

→ $\binom{k}{2}$ possible color combos

→ this is the minimum number of possible edges

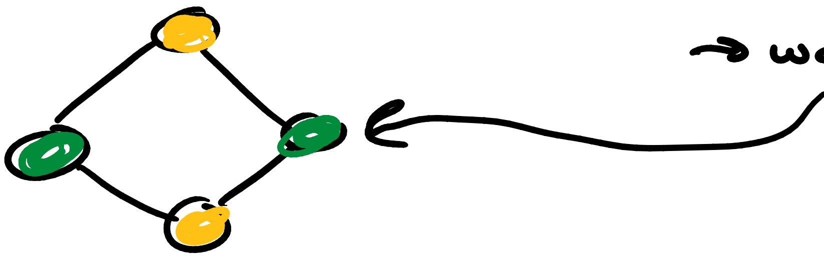
Why? Every combination must exist, as otherwise we could combine colors to get a $(k-1)$-coloring





→ no show

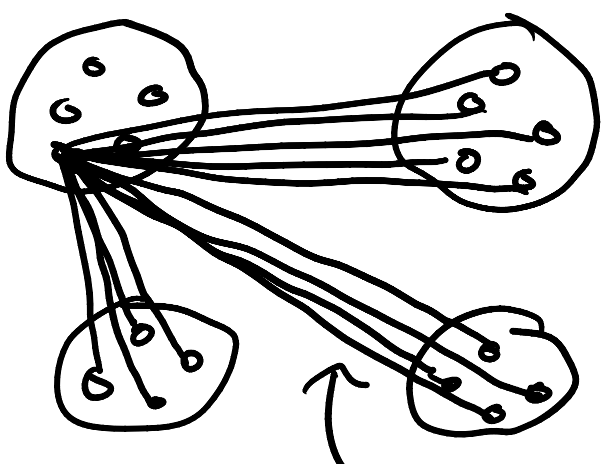
→ we can combine



⇒ any k-chromatic graph has at least $\binom{k}{2}$ edges

What about how BIG?

- consider a multi-partite graph (k-partite)



To maximize edges: make it complete

repeat for all vertices

How do we further maximize for a given k, $|V(G)|$?

→ set all partite sets equal in size $+/-$ 1 vertex

⇒ Turán graph

Q: Does it maximize $|E(G)|$

- consider an "unbalanced" multi-partite complete
- $\exists S_i, S_j$ s.t. $|S_i| + 1 < |S_j|$
- consider moving $v \in S_j$ to S_i

What does this do to $|E(G)|$?

- edges lost $= |S_i|$
- edges gained $= |S_j| - 1$

→ as $|S_j| > |S_i| + 1$ we have a net gain

$\rightarrow$ If we repeat, we will eventually maximize $|E(G)|$

$\Rightarrow$ Turán graph is the largest possible k-chromatic graph on $|V(G)|$ vertices. $\square$

Color-critical graphs

G is color-critical if

$\forall v \in V(G) \rightarrow \chi(G-v) < \chi(G)$

★ $\forall e \in E(G) \rightarrow \chi(G-e) < \chi(G)$

For color-critical graph G

$\exists$ some k-coloring on G s.t.

$\forall v \in V(G)$ the color $C(v)$ appears nowhere else and there is $k-1$ colors in $N(v)$

$\rightarrow$ Consider $(k-1)$-coloring on $G-v$

- if we add back v and not all $k-1$ colors show up in $N(v)$

all $k-1$ colors show up in $N(v)$

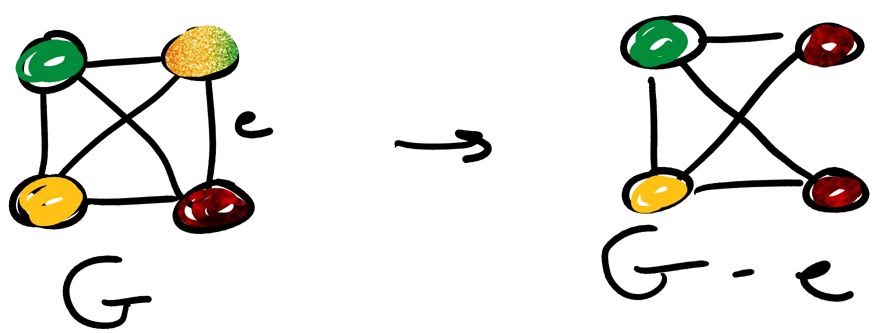
→ we have a $k-1$ coloring on G, we can just assign a color not on $N(v)$

Contradiction

Similarly: $\forall e = (u, v) \in E(G)$

- Every proper $(k-1)$-coloring of $G - e$ gives $c(u) = c(v)$

→ If not, that would equivalently be a $(k-1)$-coloring on G



Connectivity of a k-color-critical graph G

Show: G is $(k-1)$-edge-connected

To do so, first show:

For G' s.t. $\chi(G') > k$, let $\{X, Y\}$ be some partition of $V(G)$

If $G'[X]$ and $G'[Y]$ are both k-colorable $\rightarrow |[X,Y]| \geq k$

induced subgraph on X

- consider $X_1 X_2 X_3 \dots X_k$ and $Y_1 Y_2 \dots Y_k$ as independent sets defined by our assumed k-coloring

(multiple)

Show: if $|[X,Y]| < k$, $\exists X_i Y_j$ that we can combine to form a k-coloring on G'

- assume $|[X,Y]| < k$
- construct H as bigraph

$\longrightarrow$ condense each of X_i, Y_i

→ condense each of X_i, Y_i
into a single vertex

→ add edge (X_i, Y_j) if no
edge $(x,y) \in E(G') : x \in X_i, y \in Y_j$
exists on G'

Note: H has more than $k(k-1)$ edges

→ k^2 possible, but cut $< k$

Note x2: m vertices cover at
most $k*m$ edges in H

→ $E(H)$ is not covered by
only $(k-1)$ vertices

=> min cover $\geq k$

max match $\leq k$

min cover = max match = k

If we combine ALL matched
sets into a single color

=> we get a k-coloring on G'

$\Rightarrow$ we get a k-coloring of G

× × ×

Contradiction

× × × ×

$\Rightarrow |[X,Y]| \geq k$

Bring it on home: 

Every k-color-critical graph is $(k-1)$-edge-connected

- Consider k-color-critical G
- $[X,Y]$ defines some min cut

$\rightarrow G[X], G[Y]$ are $(k-1)$-colorable

$\Rightarrow$ edge cut must be at least $(k-1)$ in size $\square$

Minimum vertex coloring

- coloring G with $\chi(G)$ colors

Issue: NP-hard/complete

Takeaway: impossible to solve exactly in the general case

=> Heuristics and approximations are the name of the game

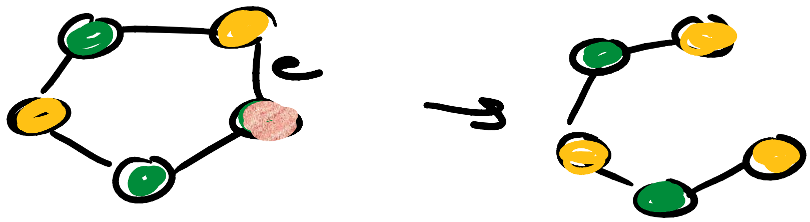
usually based on the greedy coloring algorithm and vertex processing order

Brélaz: go in order of which vertex currently has the most colors in its neighborhood

→ Logic: color "most difficult" vertices first

Optimal ordering exists → show in HW 4

From Q9: big takeaway is greedy coloring is Hugely dependent on processing order



$\chi(C_5) = 3$