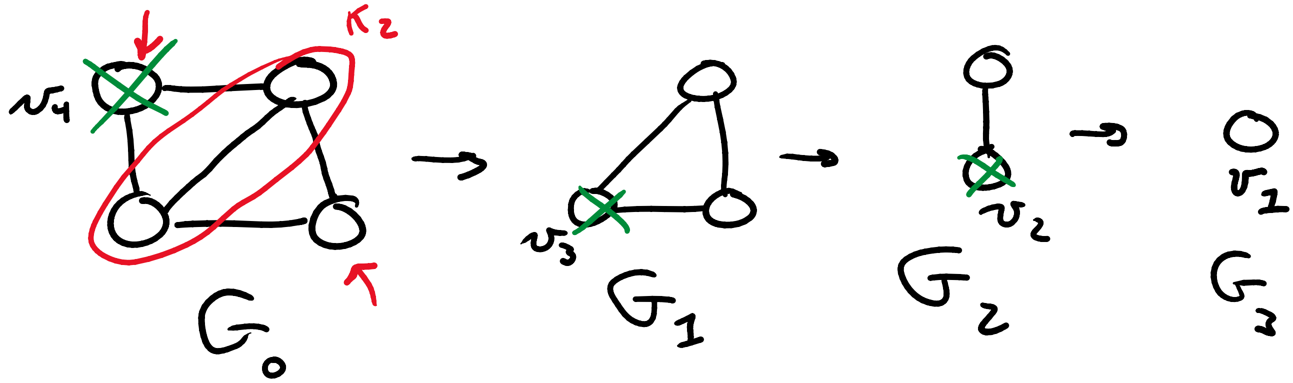
Lecture 21- Planarity

Thursday, April 7, 2022 10:00 AM

Question of the day:

Is it snowing in Vermont?

Simplicial Elimination ordering (SEO)



working backwards

$$\chi(G,k) = \prod_{i=1}^{n} (k - d'(v_i))$$

$$\chi(G,k) = (k)(k-1)(k-2)(k-2)$$

$$\chi(G,1) = 0$$

$$\chi(G,2) = 0$$

$$\chi(G,3) = 6$$

Note: max clique in G is K_3

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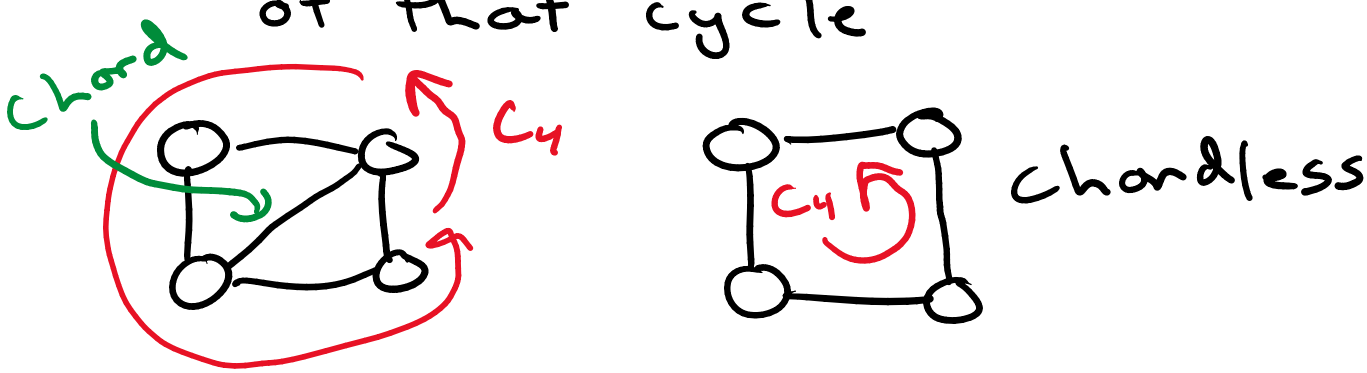
which graphs have a

Q: which graphs have a simplicial elimination ordering?

A: chordal graphs

Chordal graph: a simple graph that has no chordless cycle

Chord: an edge with endpoints on a cycle but is not part of that cycle



Chordless cycle: a cycle of length at least 4 with no chords

G has SEO $\iff$ G is chordal

G has SEO $\implies$ G is chordal

- consider some $C_{n \geq 4} \subseteq G$

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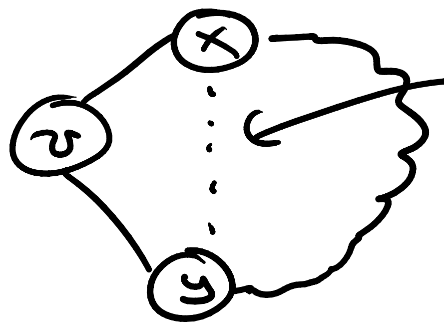
- consider the first vertex v deleted in that cycle



note: if $(x,y) \notin E(G)$ v would not be simplicial

G is chordal $\Rightarrow$ G has a SEO

Show: every chordal graph has a simplicial vertex farthest from some vertex x

Note: deleting a vertex can't introduce a chordless cycle

strong induction on $|V(G)|$

Basis $P(1)$: single vertex is simplicial

$P(n > k)$: Consider G, $|V(G)| = n$

- consider $x \in V(G)$

Case 1: $N(x) = \{V(G) - x\}$

→ $G - x$ is chordal

→ I. H. on $G - x$

→ Any simplicial vertex in $G-x$ is simplicial in G

$SEO(G) = \{x\} + SEO(G-x)$

Case 2: $N(x) \neq \{V(G) - x\}$

- define T = vertices maximum distance from x
- define H = subgraph induced by $G[T]$
- define S = vertices in $G - T$ with neighbors in $V(H) \cup T$
- define Q = component of $G - S$ that contains x

Note: S must be cliques

→ has neighbors in Q and H for all $u \in S$

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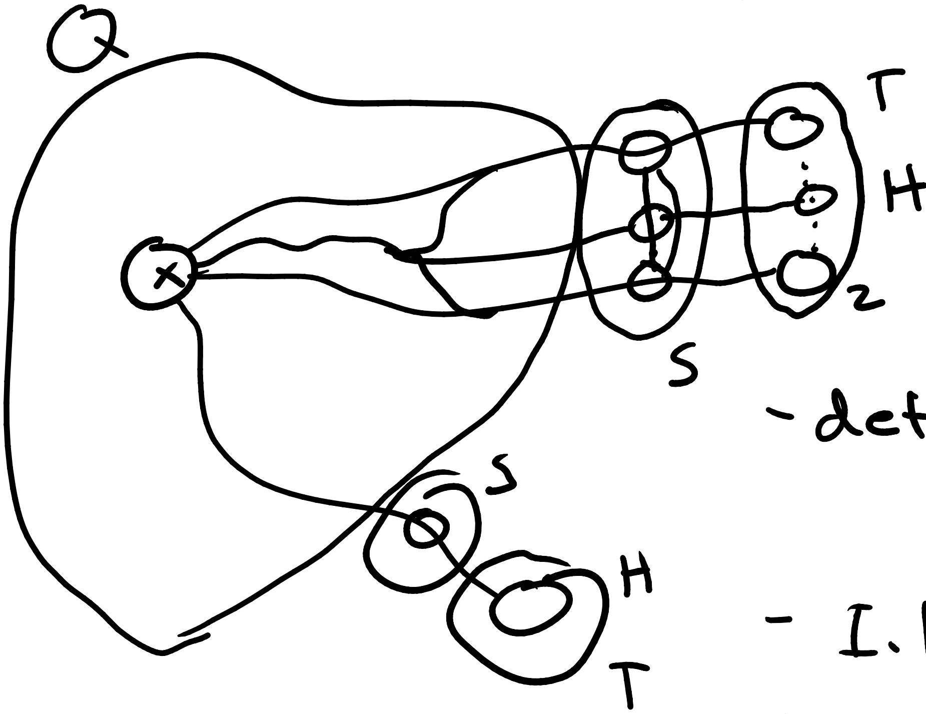
for all $w \in S$

→ any cycle from $H \to Q \to H$
passing through any $u, v \in S$
must have (u,v) as a chord



- define $G' =$
$G[S \cup V(H)]$

- I.H. on G'

→ consider some
$u \in S$

→ $\exists$ simplicial vertex
in $H \to z$

→ also simplicial in G

$\Rightarrow$ so we can construct a SEO
using this vertex z $\square$

Recall G is perfect if

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$$\chi(H) = \omega(H) : \forall H \leq G$$

$\uparrow$ clique number aka max clique size

Show: chordal graphs are perfect

we know deleting vertices cannot create a chordless cycle

$\rightarrow$ show $\chi(G) = \omega(G)$ for all chordal graphs

we know chordal G has SEO

- consider the reverse order
- as v_i is added back to G_i $N'(v_i]$ is a clique
- can use greedy coloring on each v_i when we add them back

$\rightarrow$ If v_i gets color k, v_i is in clique K_k

$\rightarrow$ etc.

→ we'll end up with some v_j with color l in clique of size l, where l is maximum over all of G

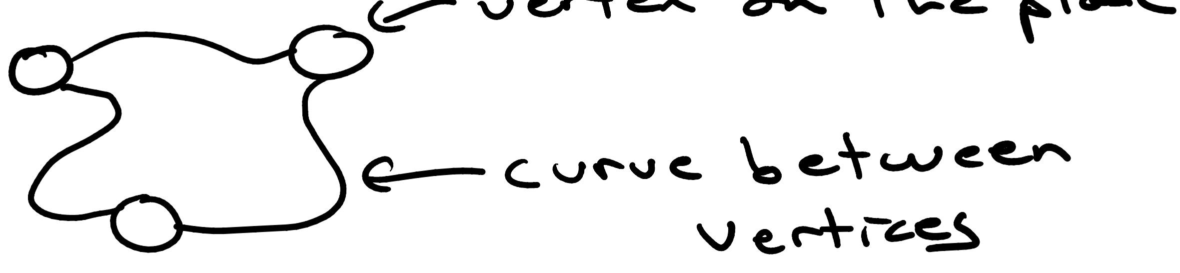
$\Rightarrow \chi(G) = \omega(G)$

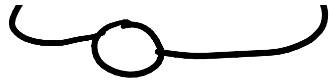
$\Rightarrow$ we can apply this same logic to all induced subgraphs $H \subseteq G$

Hence, chordal graphs are perfect $\square$

Planarity and graph drawing

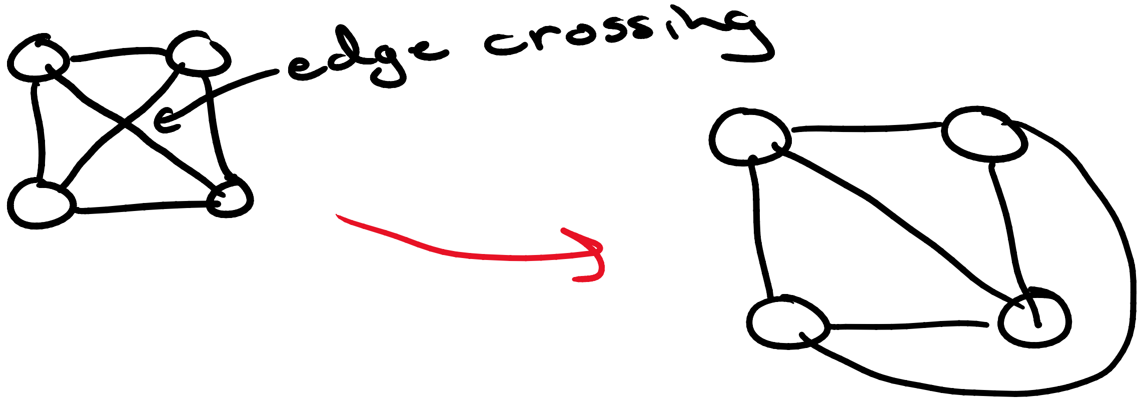
Graph drawing: a mapping of vertices to some points on the plane, and edges to some curves between those points





vertices

Graph planarity: a graph is planar if it can be drawn without any edge crossing - aka curves of edges intersect



no crossing

Couple more definitions: → planar

planar embedding → graph drawing w/o crossing

Face: maximal area in same planar embedding enclosed by edges

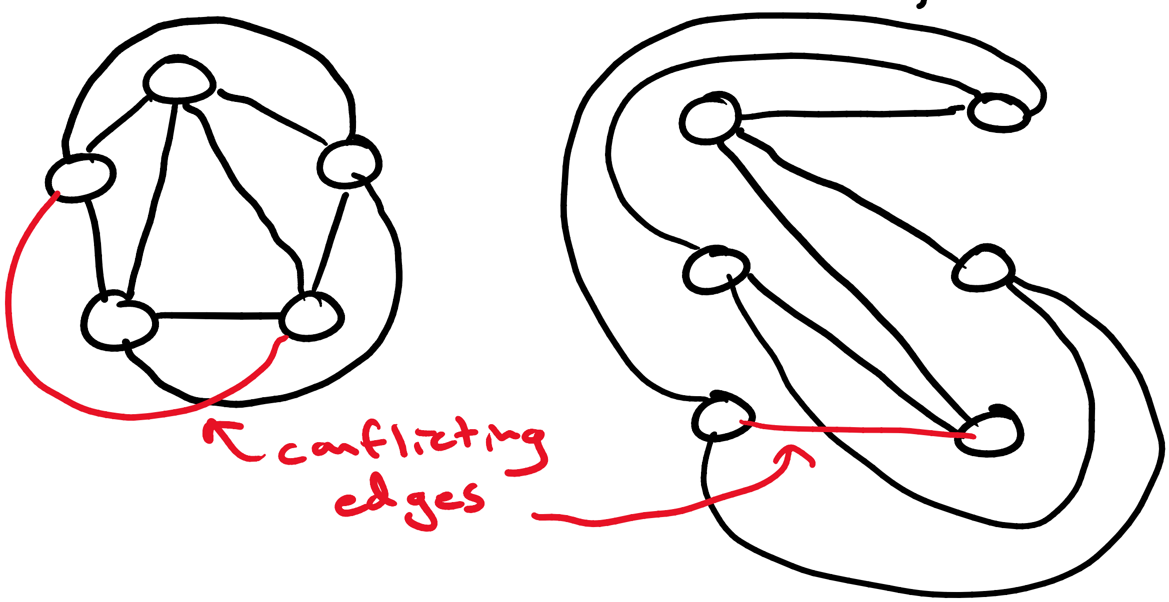
Outer Face: the external/unbounded face of the embedding

we have 4 faces of K_4

What about non-planar graphs?
aka graphs that have no planar embedding possible

What about K_5?

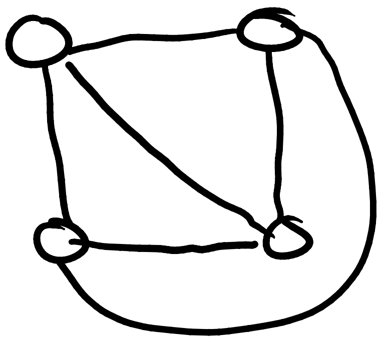
or $K_{3,3}$



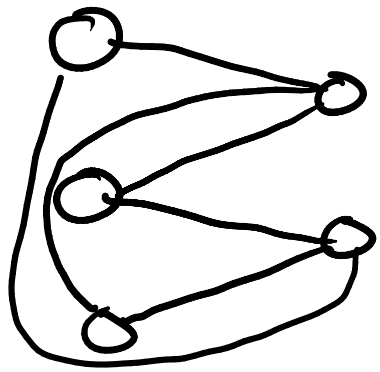
outerplanar graph: a graph with a planar embedding where all edges are on the outer face

K_4 is not outerplanar

$K_{3,3}$ is not



$K_{3,2}$ is not outerplanar



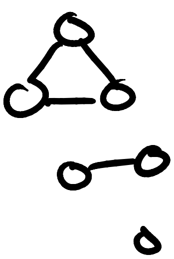
Don't worry: we'll formalize all this later

Couple more examples of planar/outerplanar

K_5 not planar

K_4 not outplanar but is planar

cycles are planar and outerplanar

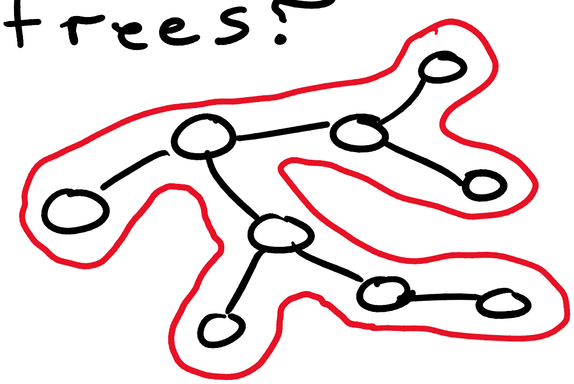


K_3 is both

K_2 " "

K_1 " "

trees?

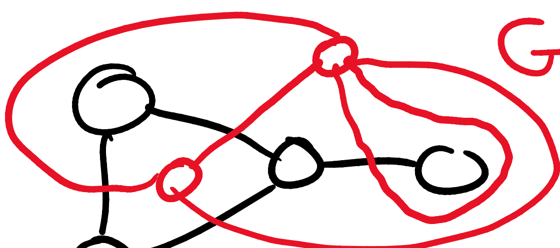


both planar and outerplanar

outerplanar

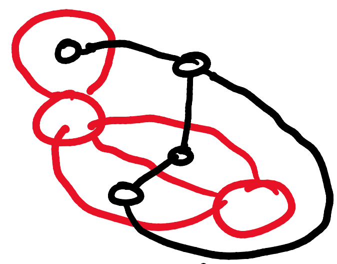
Dual Graphs

Dual graph G^* of planar embedding of G is a graph whose vertices are the faces of G and whose edges are defined based on the faces of G that share an edge



G^*

G



$(G^*)^* \cong G$

Note: the dual graph of a dual can be but is not always isomorphic to the original graph (O.G.)