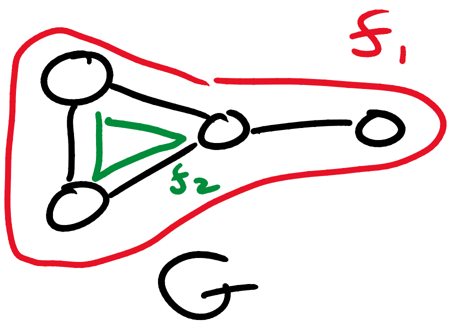
Lecture 22 - Kuratowski's Theorem

Monday, April 11, 2022 9:56 AM

Faces of planar graphs



G has two faces

→ Length of a face is the number of edges

$\ell(f_1) = 5$

$\ell(f_2) = 3$

Note: each edge contributes +2 to the sum of face lengths

$$\sum_i \ell(f_i) = 2|E(G)|$$

G is bipartite $\Leftarrow\Rightarrow$ all faces of G are even

$\Leftarrow\Rightarrow G^*$ is Eulerian

G is bipartite $\Rightarrow$ all face are even

Note: all possible closed walks on G are even on G are even

→ face length defined by a closed walk

⇒ all faces are even

all faces even ⇒ G is bipartite

- Consider some cycle in G
- The rest of G is internal or external to cycle C

- consider the internal portion of G

→ all faces are even

⇒ $\sum \ell(f_i)$ = even

Note: each internal edge will be counted twice

Note x2: each edge on C will be counted once

PARITY ⇒ the cycle of C

PARITY $\Rightarrow$ the cycle of C must be even, regarless of choice of C $\Rightarrow$ G is bipartite $\square$

All faces even $\Leftrightarrow$ G^* is Eulerian

Note: vertex degrees in G^* are the length of each face in G

$\Rightarrow$ all even $\Rightarrow$ Eulerian

($\Leftarrow$) Eulerian implies even degrees in G^* $\rightarrow$ even face lengths in G $\square$

Euler's formula

$$\underset{|V(G)|}{n} - \underset{|E(G)|}{e} + \underset{\text{\#faces in embedding of } G}{f} = 2$$

For connected planar graph G

Let us prove this using the

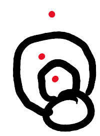
POWER

of induction on n

Basis $P(1)$:  $n=1$ $\quad 1+1=2$ ✓

$f=1$

$e=0$

 $n=1$

$f=e+1$ $\quad 1-\cancel{e+e}+1$

$e=e$ $\quad 1+1=2$ ✓

Consider our $P(n)$ case

- Note: there exists some edge which is not a self loop
- contract that edge to get our $P(k)$

I.H. on $P(k)$

$\hookrightarrow n'-e'+f'=2$

Bring it on back to $P(n)$

$n=n'+1$

$e=e'+1$

faces

$e = e' + 1$

$f = f'$ ← # faces unchanged

plug n' chug

$(n - \cancel{1}) - (e - \cancel{1}) + f = 2$

$n - e + f = 2$ $\boxed{\checkmark}$

Let's use this formula

If G is a simple connected planar graph with $|V(G)| \geq 3$, then $e \leq 3n - 6$

G is simple $\rightarrow \ell(f_i) \geq 3$

From our face length sum formula

$2e = \sum \ell(f_i) \geq 3f$ ← # faces

$2e \geq 3f$

Consider $n - e + f = 2$

$e = n + f - 2$

$3e = 3n + 3f - 6$

$$3e = 3n + 3f - 6$$

$$\cancel{3}e \le 3n + \cancel{2e} - 6$$

$$\boxed{e \le 3n - 6}$$

what if G is triangle-free?

$$\ell(f_i) \ge 4$$

$$\sum \ell(f_i) \ge 4f$$

plug n' chug

$$\boxed{e \le 2n - 4}$$

Note: if these inequalities do not hold
=> G is not planar

However => these are necessary but NOT sufficient

maximal planar G: adding an edge makes G nonplanar makes G nonplanar

minimal nonplanar G: deleting an edge
makes G planar

triangulation: a planar embedding
where all faces are
of length 3

$\boxed{\text{maximal planar} \iff \text{triangulation}}$

Necessary conditions for planarity

$e \le 3n - 6$ in general

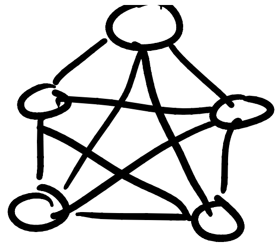
$e \le 2n - 4$ if G is triangle-free

G has no K_5 subgraph or subdivision

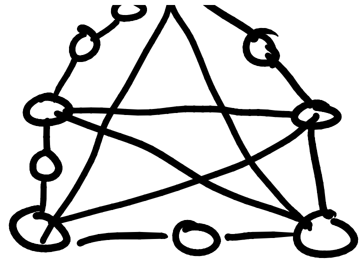
G has no $K_{3,3}$ subgraph

Note: subdivided edge

subdivision of a subgraph



K_5



K_5 subdivision

→ If H is nonplanar, then an H subdivision is nonplanar

K_5 and $K_{3,3}$ subdivisions
→ Kuratowski subgraphs

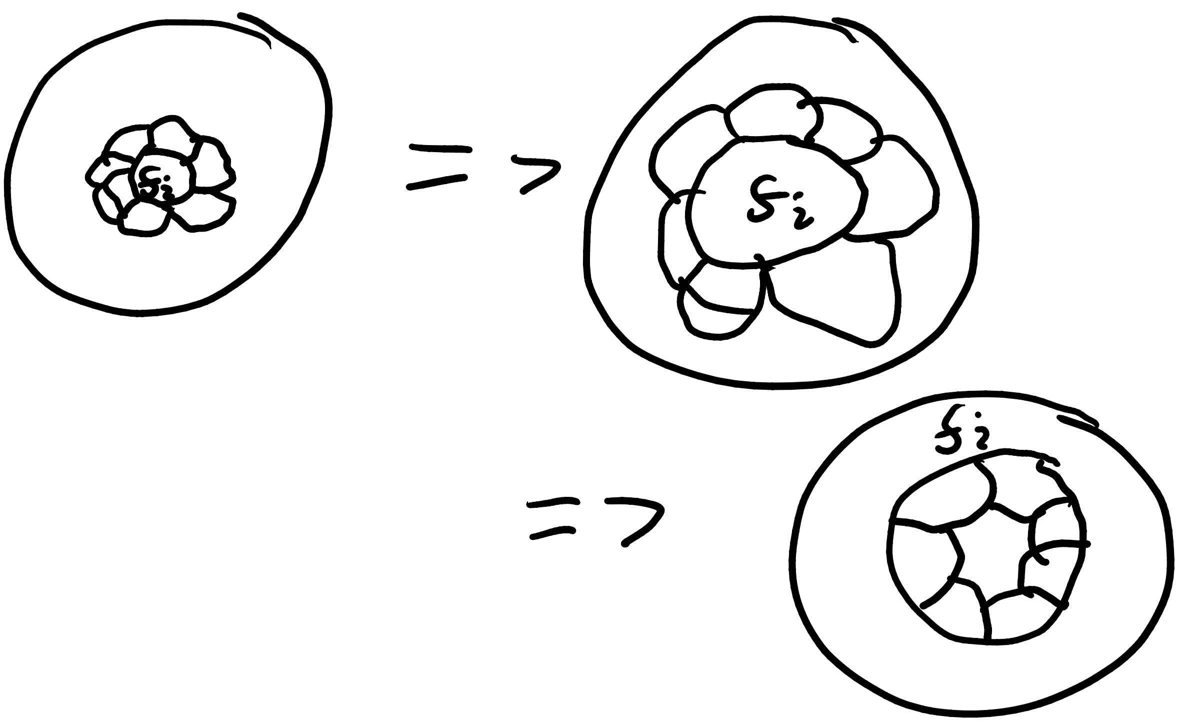
Kuratowski's Theorem

G is planar iff
G has no K.S.
(Kuratowski subgraph)

① For every face f_i of a planar embedding of G $\exists$ a planar embedding where f_i is the outer face

- Consider some embedding of G on a sphere

- Consider face f_i and return a projection of the embedding bounded by f_i



② Every minimal nonplanar graph G is 2-connected

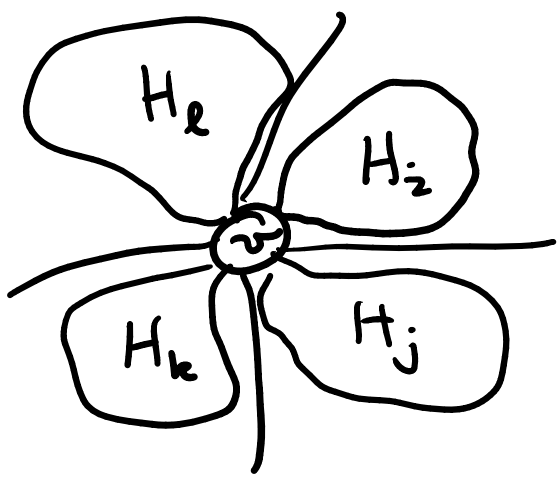
- $\forall H \subseteq G$, where $|H| < |G|$, H is planar

- Assume $\exists v \in V(G)$ s.t. $G - v$ is disconnected into

$G - v$ is disconnected into $H_1 H_2 \dots H_k$ (all planar)

- Note: we can create an embedding of G (contradiction) by "squeezing" all of H_i into $\frac{360}{k}$ degrees around v

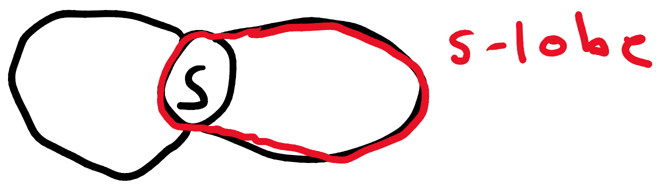
Note x2: From ①, for all H_i there exists an embedding with v on the outer face



contradiction

→ G must be 2-connected

③ S-lobe: an induced subgraph of vertex set S and some component of $G - S$



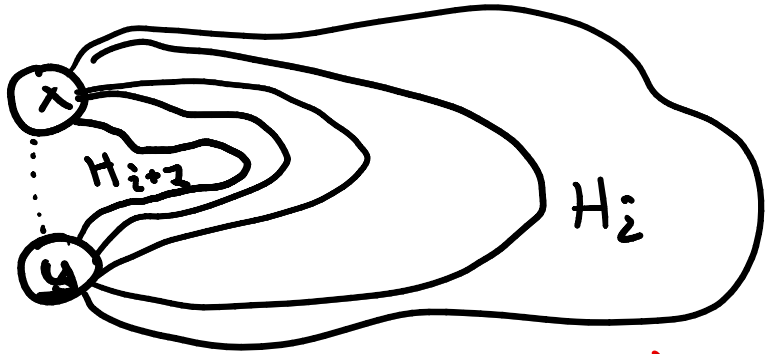
Let $S = \{x, y\}$ be a separating set of 2-connected G. If G is nonplanar $\Rightarrow$ adding edge (x, y) to some S-lobe of G yields a nonplanar graph.

- Let $H_i = \underbrace{G_i}_{\text{component of } G-S} \cup \underbrace{\{x, y\}}_{S} \cup \underbrace{(x, y)}_{\text{edge}}$

- If H_i is planar, from (1) it has an embedding where (x, y) is on the outer face

- Assume all possible H_i are planar
  $\rightarrow$ we can iteratively embed all $H_{i=1 \ldots k}$ into the face of H_{i-1} containg (x, y)

of H_{i-1} containg $\{x, y\}$



x x x
Contradiction x
x x

$\Rightarrow$ at least one H_i is nonplanar $\square$

(4) If G is a graph with the fewest edges among all nonplanar graphs without a K.S., then G is 3-connected

Note: G doesn't exist, but if it did, it would need to be 3-connected

Why → restrict any possible counter-example to 3-connected graphs

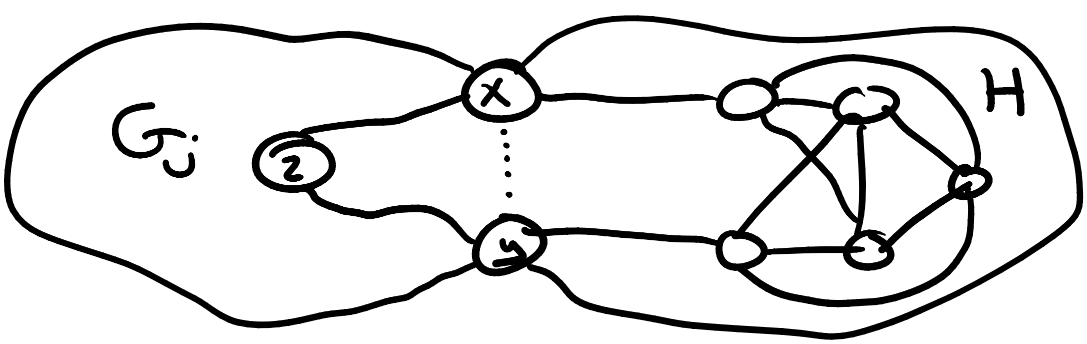
Note: deleting an edge cannot
create a K.S.

Note: deleting an edge cannot create a K.S.

→ $G - e$ is planar and does not have a K.S.

From 2, G is 2-connected

- Assume $\exists S = \{x, y\}$, then some S-lobe of $G_i + S$ is nonplanar from 3
- define H as that S-lobe $+ (x, y)$
- From our minimality condition, H must have a K.S.
- consider our current configuration



However, note that since G_i is 2-connected $\exists z \in G_j$ where we have disjoint x,z and y,z paths

$\Rightarrow$ we still have a K.S. when we consider these paths

~~Contradiction~~

$\Rightarrow$ Any counter-example must be 3-connected

counter-example: a nonplanar graph with no K.S.

Next up: show all 3-connected graphs w/o a K.S. are planar

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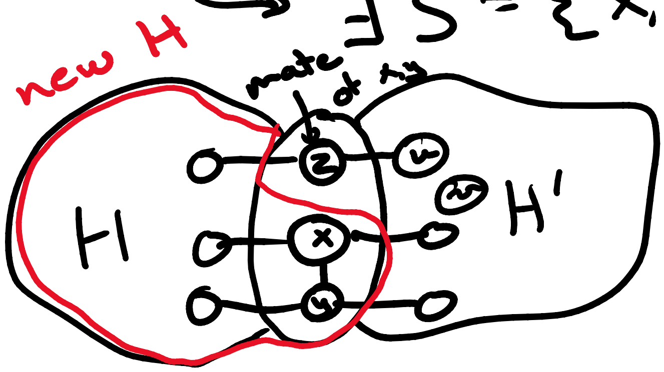
5 If G is 3-connected and $|V(G)| \geq 5$, $\exists e \in E(G)$ s.t. $G \cdot e$ is 3-connected

Consider $e = (x,y) \in E(G)$ s.t. $G \cdot e$ is not 3-connected

$G \cdot e$ is not 3-connected

$\rightarrow \exists S = \{x, y, z\}$



- Assume $\exists$ no such edge s.t. $G \cdot e$ is 3-connected

$\rightarrow$ all edges are within a separator with same 'mate' vertex

Choose $S = \{x, y, z\}$ s.t.

$|V(H)|$ is maximum

⋆ Extremal argument ⋆

Each of x, y, and z have a neighbor in each of H and H'

- consider $u \in N(z)$, $u \in V(H')$
- consider v, the mate of (u, z)

Note: $G - \{z, u, v\}$ is disconnected

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$V(H) \cup \{x, y\}$ is connected

and within only one component

of $G - \{z, u, v\}$

$|V(H) \cup \{x, y\}| > |V(H)|$

contradiction

of our selection of H

$\Rightarrow \exists e \in E(G)$ s.t.

$G \bullet e$ is 3-connected $\square$

⑥ If G has no K.S. $\Rightarrow$ $G \bullet e$ has no K.S.

contrapositive

$G \bullet e$ has K.S. $\Rightarrow$ G has K.S.

- define H as K. S. $\in G \circ e$

$\leftarrow$ contracted edge

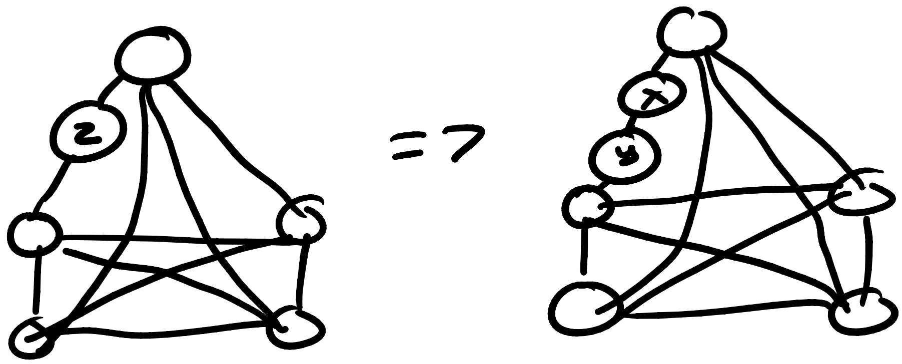
- define $z \in V(G \circ e)$, $z \leftarrow e = (x, y)$

Case 1: $z \notin H$

trivially holds

Case 2: $d(z) < 3$ (degree in H)

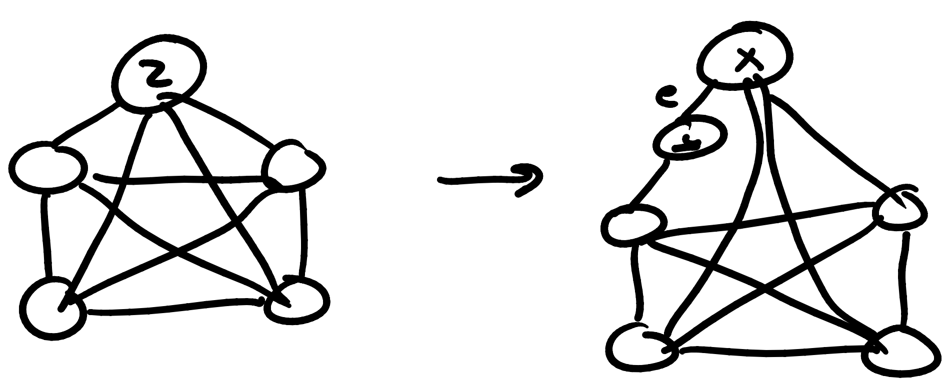
$\rightarrow$ z is along a subdivided edge



case 3: $d(z) \geq 3$

$d(x) \leq 2$ or $d(y) \leq 2$

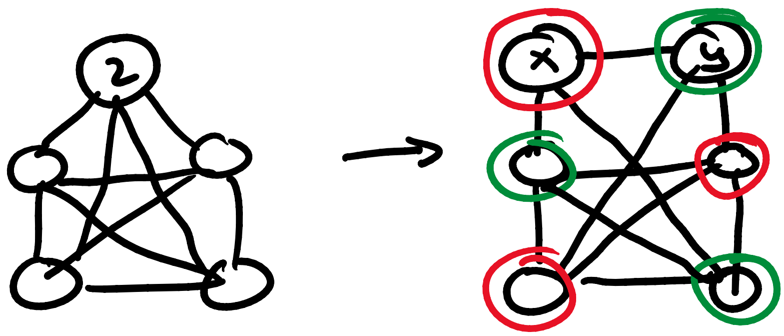
$\rightarrow$ same thing, e is along a subdivided edge



Case 4: $d(z) \geq 3$ $\quad$ $d(x) \geq 3$

$d(y) \geq 3$

$K_5 \to K_{3,3}$ is the only way



we have $K_{3,3}$ embedded