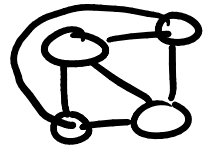
Lecture 23 - Coloring of planar graphs

Thursday, April 14, 2022 9:59 AM

Kuratowski continued:

7 If G is 3-connected with no K.S., G has an embedding on the plane

Induction on $|V(G)|$

Basis: $P(4) \Rightarrow K_4$ 

Consider our $P(n) \overset{=G}{}$ case

- Note: $\exists e = (x,y)$ s.t. $G \cdot e$ is 3-connected 5

- Note x2: if G has no K.S. then $G \cdot e$ has no K.S. 6

Get $P(k < n)$ case via $G \cdot e$

$P(k)$ has no K.S. and is 3-connected

→ I.H. on $P(k)$

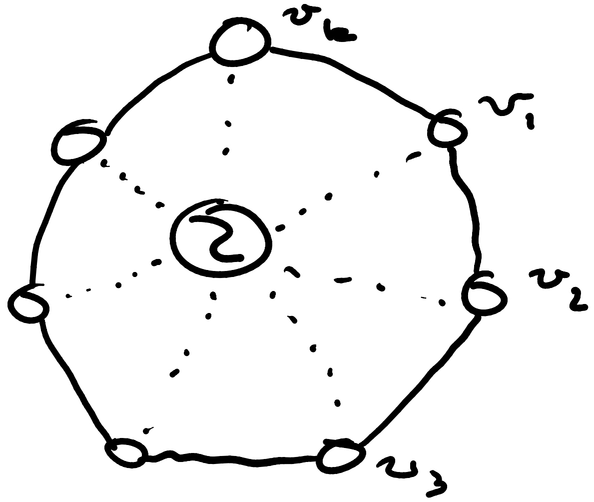
↳ $P(k)$ has an embedding on the plane

on the plane

To bring it back $P(n)$

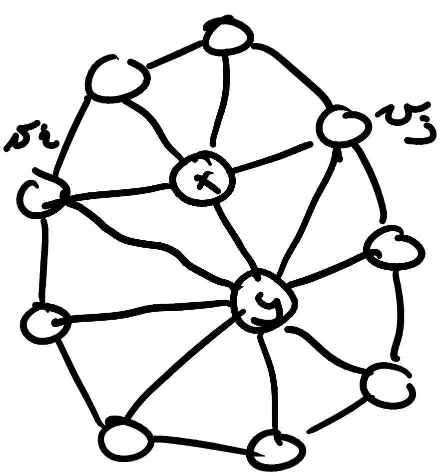
- consider $z \leftarrow (x, y) = e$

Note: all $N(z)$ form a face that contains z



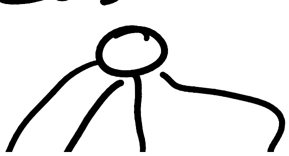
- order all $N(z)$ as $v_1 v_2 \dots v_k$ around z
- consider the following possible cases when we expand $z \to x, y$

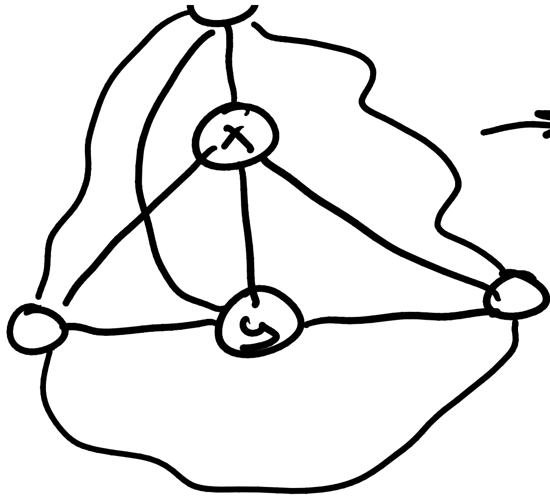
Case 1: $N(x)$ is some exclusive subset of $v_i \dots v_j$ and $|N(x) \cap N(y)| \leq 2$



$\rightarrow$ trivial to construct an embedding for x, y

Case 2: $|N(x) \cap N(y)| \geq 3$





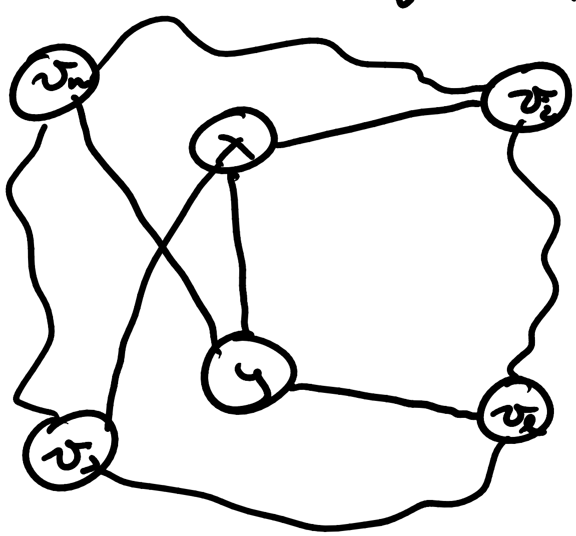
→ we have K_5 K.S.

contradiction

Case 3: $N(x)$ alternates with $N(y)$ s.t. $v_i v_j \in N(x)$

$v_e v_m \in N(y)$

$$v_i < v_e < v_j < v_m$$



→ we have $K_{3,3}$ K.S.

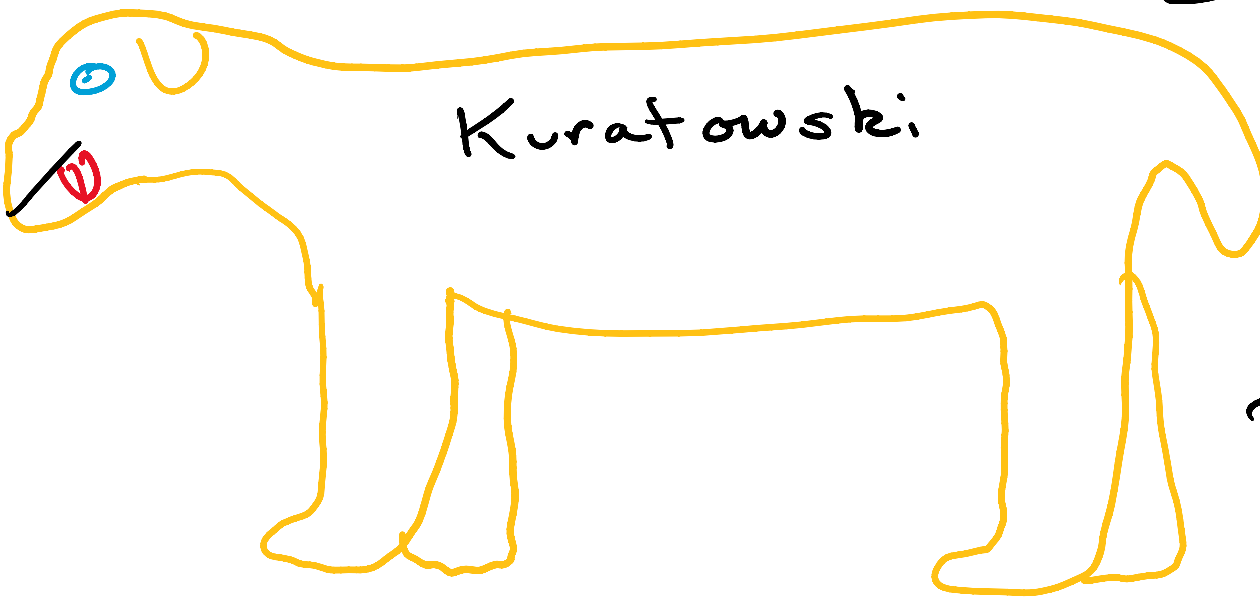
contradiction

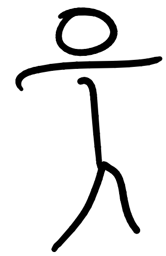
④ + ⑦ = Kuratowski's theorem

④ ↓ counter-example must be 3-connected

⑦ → no 3-connected counter-example exists

must be
3-connected

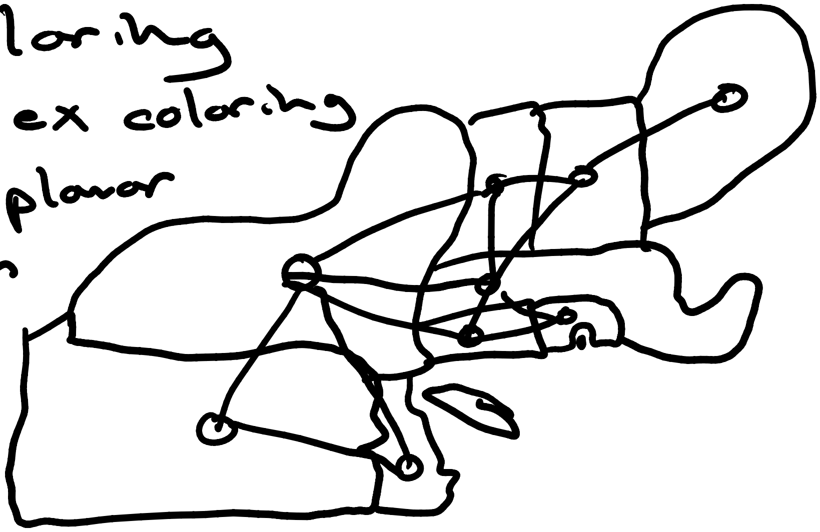




4 color theorem

Coloring of planar

Map coloring
= vertex coloring
of a planar
graph



Q: How many colors do we need?

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subQ: Can we bound $\chi(G) \le 5$ for G is planar

5-color theorem: yes we can

Approach: find a minimum counter-example

Note: such a counter-example must have some $v : d(v) \le 5$

↳ consider $m \le 3n - 6$

and $\sum d(v) = 2m$

$\hookrightarrow$ consider $m = 3n - 6$

and $\sum d(v) = 2m$

if $d(v) = 6$

$2m = 6n$

$m \leq 3n - 6$

$2m \leq 6n - 12$

$6n \leq 6n - 12$ ✗

$\Rightarrow \exists v : d(v) \leq 5$ in a planar graph

5-color theorem: all planar graphs can be colored with 5 colors

Induction on $|V(G)|$

Basis $P(\leq 5) \rightarrow$ trivial to color

$P(n)$: consider planar G

$\exists v \in V(G) : d(v) \leq 5$

$P(k)$: we take $G - v$

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→ deleting a vertex cannot create a K_5 → $P(k)$ is planar

I.H. on $P(k)$ → $P(k)$ can be colored with 5 colors

To bring it back: add back v and color it with $1 \ldots 5$

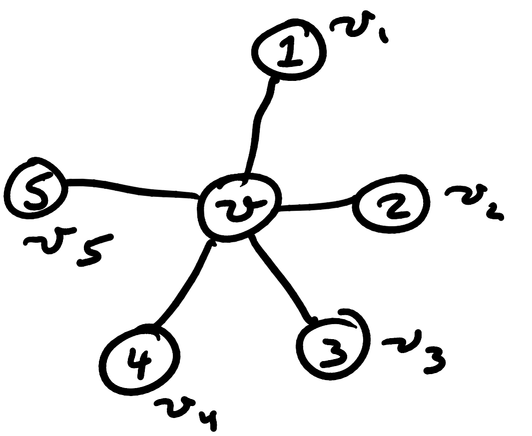
Case 1: $d(v) \leq 4$

→ trivial to color

Case 2: $d(v) = 5$ and not all 5 colors show up in $N(v)$ on $P(k)$

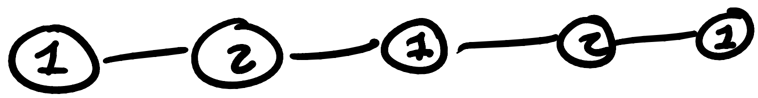
→ trivial to color

Case 3: $d(v) = 5$ all $N(v)$ will have a different color

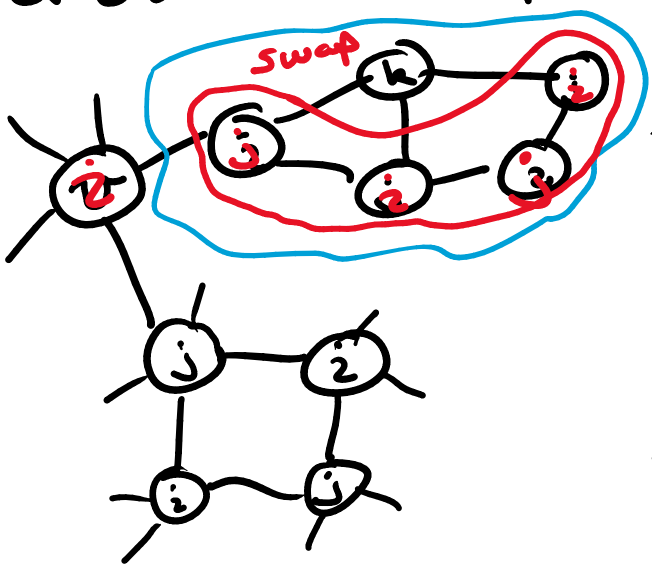


Show: is that this can be "reduced" s.t. v can be properly colored w/ $1 \ldots 5$

Kempe chains: color-alternating paths



Consider all possible Kempe chains around v for i, j color pairs

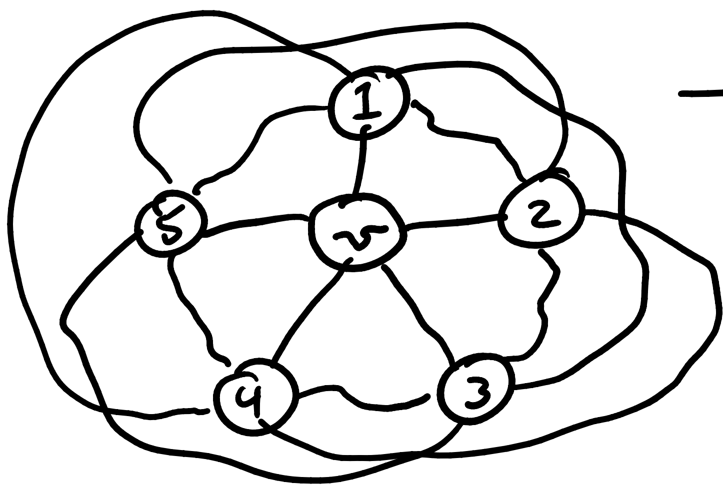


* If for a given i, j pair of colors, any ij-alternating paths don't intersect

→ we can swap all i, j colors on one induced subgraph of Kempe chains

AND color v with the color no long in $N(v)$

Q: will there always be such an indepent i,j pair such an indepent i,j pair of Kempe chain colors



→ Note: if these paths exist, we have a K_5 K.S.

contradiction

$\Rightarrow \exists$ at least one i,j pairs that we can reduces s.t. we can properly 5-color G $\square$

What about 4 colors?

Can we do the same approach?

As we saw: we're looking to find some minimum unavoidable configuration that a counter-example must have

→ If we can show all configuration are reducible to color v with are reducible to color v with some color 1...4

$\Rightarrow$ all planar graphs are 4-colorable

To minimize possible configurations

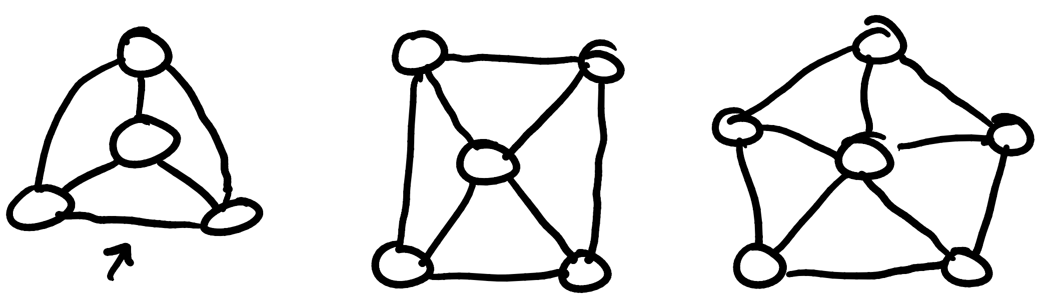
→ consider a triangulation

(all planar graphs are a subgraph of a triangulation)

And if all triangulations are 4-colorable → all subgraphs are too

→ aka all planar G

From the 5-color theorem proof: possible configurations



Note: $\delta(G) \geq 3$ if G is a triangulation

Let's try again for 4 colors

Let's try again for 4 colors

Basis $P(\leq 4) \rightarrow$ trivial

$P(n)$: we have G, $v \in V(G)$

s.t. $d(v) \leq 5$

$P(k)$: $G - v$

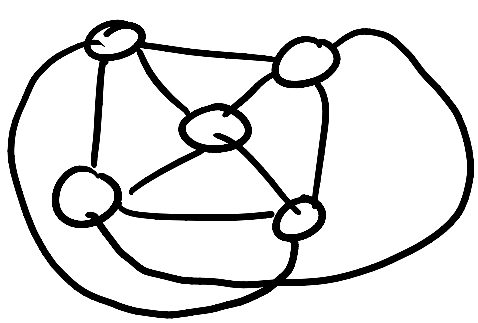
I.H. $\rightarrow P(k)$ is 4-colorable

Let's bring it on back now:

Case 1: $d(v) = 3 \rightarrow$ trivial

Case 2: $d(v) = 4$

we can use same argument as our 5 color theorem



$\rightarrow K_5$

Case 3: $d(v) = 5$

Note: exactly two vertices in $N(v)$ have the same color

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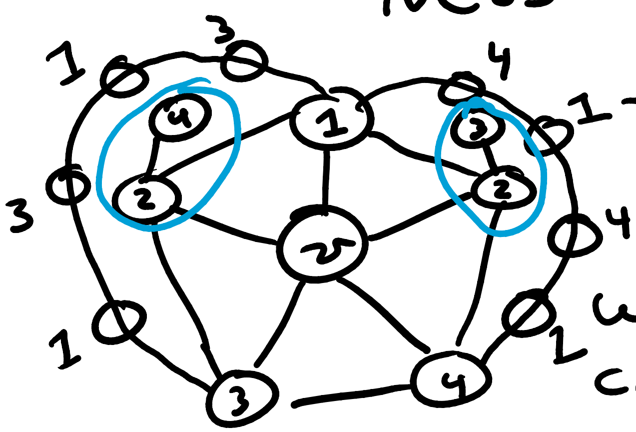


$1 \to$ consider paths from $1 \to 3$ and $1 \to 4$

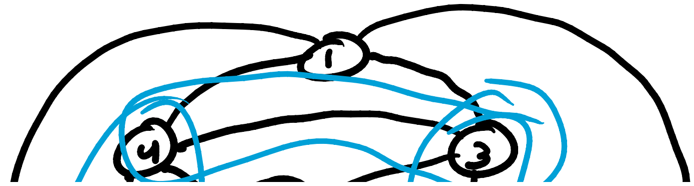
we could eliminate color 1 if these paths did not exist

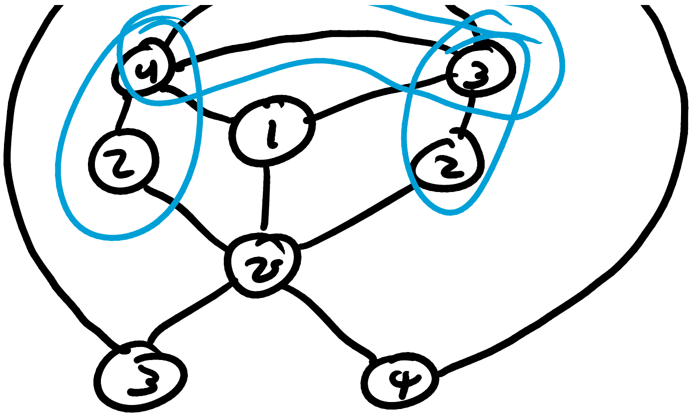
Now consider the vertices colored 2

As we can't have a 2-3 alternating path $\to$ we can swap colors of both of the induced subgraphs containing colors

$\to$ we eliminate a color 2 from $N(v)$ and can color v with color 2 $\square$

NOPE





From this: we need to consider larger possible configurations

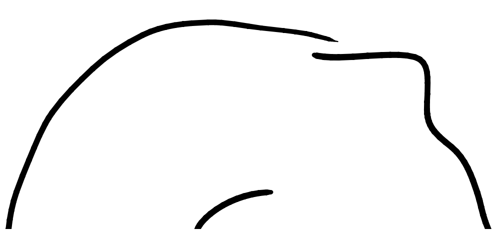
How many?

Original 4-color proof: 1800

Now: 600

Our actual 4-color theorem proof: computationally determine all possible minimal configurations and their reductions

$N(v) = V(G)$

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