## 5.1 Directed Graphs

Until today, we were only considering graphs with symmetric relations in the edges. Now, we're considering **directed graphs** or **digraphs**, where the edges have a defined directionality. The vertex where an edge starts is the **tail** and the vertex that is pointed to is the **head**. These together are the **endpoints**. We also term the tail as the **predecessor** of the head and the head as the **successor** of the tail. We can easily create a directed graph from an undirected graph by *orienting* each edge. An **orientation** of an undirected graph involves the selection of a direction for each edge, to create a directed graph.

Like with our undirected graphs. We can consider digraphs as **simple digraphs** if they don't have repeated edges or loops. Note that a simple digraph can have two edges between the same two vertices as long as they point in opposite directions. *Loopy digraphs* contain directed loops and *multi-digraphs* can contain multiple edges of the same directionality between the same two vertices.

We have similar definitions in directed graphs for **walks**, **paths**, **trails**, and **cycles**. Likewise, we have the same concepts of **subgraphs** and **isomorphism**. The **adjacency** matrix is created in a similar row-wise fashion, where a nonzero in position $(x, y)$ indicates one or more edges pointing from vertex $x$ to vertex $y$. Generally, the adjacency matrix is not guaranteed to be symmetric.

Instead of just one measure of degree degree, digraphs consider both **out degree** ($d^+(v)$) or **in degree** ($d^-(v)$). We also have the out neighborhood ($N^+(v)$) or successor set and the in neighborhood ($N^-(v)$) or predecessor set.

## 5.2 Directed Graph Degrees

For directed graphs, we've already seen that we consider both **out degree** ($d^+(v)$) or **in degree** ($d^-(v)$) separately. We likewise have minimum and maximum out and in degrees:

$$\delta^-(v), \delta^+(v), \Delta^+(v), \Delta^-(v)$$

And our degree sum formula for digraphs:

$$\sum_{v \in V(G)} d^+(v) = |E(G)| = \sum_{v \in V(G)} d^-(v)$$

As we treat the degrees of vertices in a digraph as pairs (out degree, in degree), we define the **degree sequence** for digraphs as a list of such pairs.

$$S = \{(d^+(v_1), d^-(v_1)), (d^+(v_2), d^-(v_2)), \ldots (d^+(v_n), d^-(v_n))\}$$

We have a similar notion of realizability, given the above. Let's prove that *a list of pairs of nonnegative integers is realizable as a degree sequence of a directed graph if and only if the sum of all first values in the pairs equal the sum of all second values in the pairs.* Note that here, we can consider multi-edges in our realization.

Even further, we can consider **Eulerian digraphs**. Similar to before, a digraph is Eulerian if there exists a closed directed trail containing all edges. As the proof for the directed case is identical to the undirected case, we leave it as an exercise for the reader (or as a question on a future quiz, homework, or exam).