19.1 Planarity

A **curve** is the image of a continuous map from $[0, 1]$ to $\mathbb{R}^2$. A **polygonal curve** is a curve composed of finitely many line segments. A **polygonal** u, v**-curve** starts at u and ends at v.

A **drawing** of a graph is a function f defined on defined on $V(G) \cup E(G)$ that assigns each $v \in V(G)$ to a distinct point $f(v)$ in the plane and assigns each $e = (u, v) \in E(G)$ a polygonal $f(u), f(v)$-curve. A point $x = f(e) \cap f(e')$ where $e \neq e'$ and x isn't a common endpoint of e and e' is called a **crossing**.

A graph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of G. A **plane graph** is a particular planar embedding of a planar graph. The **faces** of a plane graph are the maximal regions of the plane that contain no point in the embedding. Every finite plane graph has one unbounded face, the **outer face**.

Let's try and see if K_5 and $K_{3,3}$ are not planar; i.e., we can't draw them such that no crossing exists.

A graph is **outerplanar** if it has an embedding with every vertex on the boundary of the unbounded face. The boundary of the outer face of a 2-connected outerplanar graph is a spanning cycle.

Let's now try and see if K_4 and $K_{2,3}$ are planar but not outerplanar.

19.2 Dual Graphs

The **dual graph** G^* of a plane graph G is a plane graph whose vertices are the faces of G. An edge $e^* = (x, y) \in G^*$ connects vertices x, y representing the faces X, Y separated by an edge $e \in E(G)$. The number of edges incident to $x \in V(G^*)$ in the plane graph is the number of the edges bounding the face of X in G in a walk around its boundary.

A dual graph can be dependent on a particular embedding of a planar graph. I.e., two embeddings of a planar graph can have dual graphs that are not isomorphic. However, whenever G is connected, it is possible for us to draw the dual such that G is isomorphic to $(G^*)^*$.

The **length** of a face of a plane graph G is the total length of the closed walks in G bounding the face. If $l(F_i)$ is the length of face F_i in plane graph G, then $2|E(G)| = \sum l(F_i)$.

The following are all equivalence statements:

1. Plane graph G is bipartite.
2. Every face of G has even length.
3. The dual graph G^* of G is Eulerian.

19.3 Euler's Formula

Euler's Formula, $(n - e + f = 2)$, relates the number of vertices n with the number of edges e and faces f in a connected planar graph. We can easily prove that this relation holds with induction. This implies that all planar embeddings of a connected graph G have the same number of faces. We can also use this relation to show that if G is a simple plane graph with at least three vertices, then $e \leq 3n - 6$. If G is triangle-free, then $e \leq 2n - 4$. Additionally, we can see use the relation to more formally prove that K_5 and $K_{3,3}$ are non-planar.

A **maximal planar graph** is a simple planar graph graph that is not a spanning subgraph of another planar graph (except one isomorphic to itself). A **triangulation** is a simple plane graph where every face boundary is a 3-cycle. We can show that if G is a maximal planar graph, then G is a triangulation with $3n - 6$ edges.