

## 23.1 Hamiltonian Cycles

Recall that a **spanning cycle** is cycle in some graph that contains all vertices of that graph. We're going to go a bit more in-depth with this concept. Not all graphs have a spanning cycle. Graphs with a spanning cycle are called **Hamiltonian Graphs**. Such a cycle is commonly referred to as a **Hamiltonian Cycle**. What properties must a Hamiltonian graph have? We're going to consider simple and connected (obviously) graphs here.

- A Hamiltonian graph must be biconnected.
- A Hamiltonian bipartite graph must have equal sized sets.
- If  $c(H)$  is the number of components of a graph  $H$ , then Hamiltonian graph  $G$  must satisfy  $c(G - S) \leq |S|$  for all possible  $S \subseteq V(G), S \neq \emptyset$ .

These give us several necessary conditions. But what about sufficient conditions?

## 23.2 Sufficient Conditions for Hamiltonian Graphs

For the following conditions and discussions, again consider all graphs as simple and connected.

If  $G$  has at least three vertices and  $\delta(G) \geq \frac{n}{2}$ , then  $G$  is Hamiltonian.

If  $\forall u, v \in V(G) : (u, v) \notin E(G), d(v) + d(u) \geq n$ ,  $G$  is Hamiltonian if and only if  $G + (u, v)$  is Hamiltonian.

A **closure** of a graph  $G$ ,  $C(G)$ , is the graph with vertex set  $V(G)$  obtained from  $G$  by iteratively adding edges joining pairs of nonadjacent vertices whose degrees sum to at least  $n$ , until no such pair remains. A graph  $G$  is Hamiltonian if and only if its closure  $C(G)$  is Hamiltonian. The closure of  $G$  is also well-defined.

Consider graph  $G$  with vertex degrees  $d_1 \leq \dots \leq d_n$ , where  $n \geq 3$ . If  $i < \frac{n}{2}$  implies that  $d_i > i$  or  $d_{n-i} \geq n - i$ , then  $G$  is Hamiltonian. This would further imply that the closure of  $G$  is  $K_n$ . This is **Chvátals's Condition**. So if we were to ever determine that the closure of a graph is a clique, then we'd know that the original graph is also Hamiltonian.

A **Hamiltonian path** is a spanning path. A join between two graphs  $G$  and  $H$ ,  $G \vee H$ , is the graph created by adding edges between all vertices of  $G$  with all vertices of  $H$ . Or if  $J = G \vee H$ ,  $V(J) = V(G) \cup V(H)$  and  $E(J) = E(G) \cup E(V) \cup \{\forall u \in V(G), \forall v \in V(H) : (u, v)\}$ . A graph  $G$  has a Hamiltonian path if and only if  $G \vee K_1$  has a Hamiltonian cycle.

We can deduce a similar condition as above for spanning paths. For simple graph  $G$  with degrees  $d_1 \leq \dots \leq d_n$ , if  $i < \frac{n+1}{2}$  implies  $d_i \geq i$  or  $d_{n+1-i} \geq n - i$ , then  $G$  has a spanning path. Think of this as Chvátal's Condition applied to some graph that is joined with  $K_1$ . If the join has a spanning cycle, then the graph itself must have a spanning path.