

DEFINITIONS

Connected: $\forall u, v \in V(G) : \exists u, v$ -path

Component: maximal connected subgraph

Cut-edge: edge that when deleted disconnects some component

Cut-vertex: vertex that when deleted disconnects some component

Subgraph: subset of vertices and edges of some graph G

Induced: given vertex subset, subgraph containing those vertices and all edges between those vertices in some G

Spanning: a subgraph that contains all vertices of graph G

Decomposition: list of subgraphs such that every edge in some G appears exactly once in some subgraph in the list

Complement: (\bar{G}) Use $V(G)$ and add edges between vertices that aren't adjacent in G

Orientation: An orientation of undirected G creates a directed graph by assigning directions to all edges

Isomorphism: there exists a bijection between vertex sets of some G and G' such that edge relationship are preserved

Automorphism: there exists a bijection within vertex set of some G and G' such that the edge list is preserved

Length: usually measured in edges traversed

Distance: defined between some u, v – length of shortest u, v -path

Diameter: length of longest shortest path

Girth: length of shortest cycle

Eccentricity: for some u , length of longest shortest path from some u to any v in some graph

Center: subgraph induced on all vertices of minimum eccentricity

Degree Sequence: list of degrees for some real or hypothetical G

Graphic Sequence: a degree sequence for some simple graph G (it can *realize* G) **Degree Sum Formula:** $\sum_{v \in V(G)} d(v) = 2m$

Degree Sum Formula (digraphs): $\sum_{v \in V(G)} d^+(v) = |E(G)| = \sum_{v \in V(G)} d^-(v)$

Cayley's Formula: n^{n-2} possible trees

Match: set of non-loop edges with no shared endpoints

Saturated: vertex with matched edge incident

Perfect Match: all vertices saturated in some G

M -alternating Path: path that alternates between edges in match M and not in M

M -augmenting Path: M -alternating path that starts and ends at unsaturated vertices

Stable Matching: matches between two bipartite sets, where each member of each set has an ordered preference for potential matches to the opposite set, and there are no unstable pairs – pair of vertices x, a that are not currently matched, but both vertices have a higher preference for match (x, a) over their current match partner

Vertex Cover: a vertex set that contains at least one endpoint on all $e \in E(G)$

Edge Cover: an edge set that has as at least as one endpoint of all $v \in V(G)$

Independent Set: set of vertices on a graph G that are not connected by an edge

Dominating Set: set of vertices on a graph G such that all

vertices of G are either in S or have a neighbor in S

Separating Set: Set of vertices S on G such that $G - S$ has more components than G or is a single vertex. (AKA **vertex cut** or **vertex separator**)

Disconnecting Set: Set of edges F on G such that $G - F$ has more components than G (AKA **edge separator**)

Edge Cut: A set of edges F between a bipartitions of vertices (V_1, V_2) on G such that $F = [V_1, V_2] : V_1 \cup V_2 = V(G), V_1 \cap V_2 = \emptyset$

Open-ear Decomposition decomposition of G into a sequence of paths P_0, P_1, \dots, P_k , where P_0 is a closed path (cycle) and for $i \geq 1$ P_i has unique endpoints in $P_0 \cup \dots \cup P_{i-1}$

Closed-ear Decomposition: decomposition of G into paths P_0, \dots, P_k such that P_0 is a cycle and P_i for $i \geq 1$ is a path with unique or non-unique endpoints in $P_0 \cup \dots \cup P_{i-1}$

Feasible Flow: on s, t -flow network G with edge capacities $c(e)$ and flows $f(e) - \forall e \in E(G) : 0 \leq f(e) \leq c(e)$ and $\forall v \in V(G), v \neq s, t, f^+(v) = f^-(v)$

k -coloring: a vertex labeling of G with $f : V(G) \rightarrow S$, with $k = |S|$

Proper Coloring: no adjacent vertices in a k -coloring share the same color

Chromatic Number: $(\chi(G))$ the least k for which G is properly k -colorable

Color-critical: when every subgraph H of G has a lesser chromatic number than G

Chromatic Polynomial: $(\chi(G; k))$ the number of proper k -colorings of G

Mycielski's Construction: given triangle-free graph G with vertices $\{v_1, \dots, v_n\}$ and $\chi(G) = k$, we can create G' with $\chi(G') = k + 1$ by adding vertices $\{u_1, \dots, u_n\}$ and vertex w to G , making u_i adjacent to all in $N(v_i)$ and having $N(w) = \{u_1, \dots, u_n\}$

Chord: an edge on unique endpoints of a cycle of at least length 4 that is not part of the cycle

Simplicial Vertex: a vertex in G where its neighborhood is a clique (Note: K_0, K_1 , and K_2 all count)

Simplicial Elimination Ordering: an ordering of all vertices $\{v_n, \dots, v_1\}$ for deletion in G such that each vertex v_i is simplicial in the remaining graph induced by $\{v_i, \dots, v_1\}$

Planar Embedding: a specific drawing of a planar graph

Triangulation: A planar graph with all faces of length 3

Dual Graph: (G^*) of a plane graph G is a plane graph whose vertices are the faces of G . An edge $e^* = (x, y) \in G^*$ connects vertices x, y representing the faces X, Y separated by an edge $e \in E(G)$

S -lobe: an induced subgraph on G using vertices in separator S along with vertices in some component of $(G - S)$

Subdivision: take edge $e = (u, v)$ and split it into $f = (u, w), g = (w, v)$. An H -subdivision of some G is the graph resulting from (recursively) performing subdivision operations on some edges of G

Line Graph: $(L(G))$ is the simple graph whose vertices are the edges of G . Edges $v, u \in E(G)$, represented as vertices $v, u \in V(L(G))$, have an edge between them in $L(G)$ if they share a common endpoint $w \in V(G) : \{(v, w), (u, w)\} \in E(G)$ **Edge Coloring:** assigning labels to all $e \in E(G)$ such that no two edges have the same color if they share an endpoint $v \in V(G)$ (i.e., a **proper edge coloring**)

Hamiltonian Cycle: a spanning cycle

Hamiltonian Path: a spanning path

Closure: ($C(G)$) is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degrees sum to at least n , until no such pair remains

Erdős-Rényi: $G(n, p)$ or $G(n, m)$, defined on n vertices with m edges or attachment probability p . For $G(n, m)$, endpoints for each edge are selected randomly with replacement. For $G(n, p)$, a weighted coin is flipped for all u, v pairs of vertices to create edge with probability p

Configuration Model: Given a degree sequence, we biject degrees onto a set of vertices, creating d_i stubs for each vertex, where d_i is the assigned degree. Stubs are then randomly connected

Chung-Lu: We create edges between vertex pairs with probability $p_{u,v} = \frac{w_u w_v}{\sum_i w_i}$, with weights w_u, w_v commonly interpreted as degrees from a distribution.

PageRank: Given adjacency matrix A and diagonal degree matrix D , we compute the solution to the eigenvector problem of $p = (D^{-1}A)^T p$ using power iteration

GRAPH CLASSES

Simple: no multi-loops or self edges; simple directed graphs can have mirrored edges

Loopy: can have loops

Multigraph: can have multi-edges

Empty: no edges and some number of vertices

Trivial: no edges and a single vertex

Null: no vertices or edges

Complete: has all possible edges

Bipartite: union of two disjoint independent vertex sets

Digraph: directed graph, has directed edges

Tree: connected, undirected, acyclic (has no cycles)

Forest: undirected, acyclic

Leaf: vertex of degree-1

Path: (P_n) doesn't repeat vertices or edges

Traill: can repeat vertices but not edges

Walk: can repeat vertices and edges

Closed P/T/W: P/T/W that starts/ends at same vertex

Clique: (K_n) complete graph on n vertices

Cycle: (C_n) closed path on n vertices

Triangle: cycle of length 3

Star: (S_n) connected n -vertex graph with $n - 1$ vertices of degree-1 **Turán Graph:** complete r -partite graph with n vertices whose partite sets differ in size by at most 1

k-regular: has all vertices with degree of k

Eulerian: has closed trail containing all edges (Euler Tour)

Graceful: graph with graceful labeling – all n vertices and m edges labeled uniquely with $0 \dots m$, such that each edge has a unique value computed as absolute difference of its endpoints' labels

Chordal: a graph with no chordless cycles

Perfect: a graph G where $\chi(H) = \omega(H)$ for all possible induced subgraphs $H \subseteq G$

Planar: a graph that can be drawn on the plane such that no edges cross

Outerplanar: a graph that can be drawn with all vertices

on the outer face

Hamiltonian: a graph containing a spanning cycle

THEOREMS

Havel-Hakimi: A sequence $S = \{d_1, d_2, \dots, d_n\}$ is a graphic sequence iff sequence $S' = \{d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n\}$ is a graphic sequence, where $d_1 \geq d_2 \geq \dots \geq d_n$ and $n \geq 2$ and $d_1 \geq 1$

Berge: match M is maximum iff G has no M -augmenting paths

Hall: X, Y -bipartite graph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all possible $S \subseteq X$

Tutte: graph G with a perfect match satisfies the inequality $\forall S \subseteq V(G) : o(G - S) \leq |S|$

König-Egerváry: if G is a bipartite graph, then the size of a maximum matching in G equals the minimum size of a vertex cover

Connectivity Bounds: $\kappa(G) \leq \kappa'(G) \leq \Delta(G)$

Whitney: G is 2-connected iff $\forall u, v \in V(G) : \exists 2$ u, v -internally disjoint paths

Menger: G is k -connected iff $\forall u, v \in V(G) : \exists k$ u, v -internally disjoint paths

Max-flow Min-cut: The maximum value of a feasible flow equals the minimum capacity source-sink cut on a flow network

Brooks' Theorem: $\chi(G) \leq \Delta(G)$ if G is not an odd cycle or clique

Fundamental Reduction Theorem: $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$

Euler's Formula: $n - e + f = 2$

Kuratowski: A graph G is planar iff it contains no K_5 or $K_{3,3}$ subdivisions

Four Color Theorem: All planar graphs have a chromatic number of at most 4

Forbidden Subgraphs: A graph G is the line graph of some H iff G contains no double odd triangles or claws

Chvátal's Condition: Consider graph G with vertex degrees $d_1 \leq \dots \leq d_n$, where $n \geq 3$. If $i < \frac{n}{2}$ implies that $d_i > i$ or $d_{n-i} \geq n - i$, then G is Hamiltonian

Hamiltonian Necessity: if $c(H)$ is the number of components of a graph H , then Hamiltonian G must satisfy $c(G - S) \leq |S|$ for all possible $S \subseteq V(G), S \neq \emptyset$

ALGORITHMS

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procedure GREEDYCOLORING(Graph  $G(V, E)$ )
  for all  $v \in V(G)$  do
     $color(v) \leftarrow -1$ 
  for all  $v \in V(G)$  in order do
     $isCol(1 \dots \Delta(G) + 1) \leftarrow \text{false}$ 
    for all  $u \in N(v)$  where  $color(u) \neq -1$  do
       $isCol(color(u)) \leftarrow \text{true}$ 
    for  $k = 1 \dots \Delta(G) + 1$  do
      if  $isCol(k) = \text{false}$  then
         $color(v) \leftarrow k$ 
        break
   $k \leftarrow \max(color(1 \dots n))$ 
return  $k$ 

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