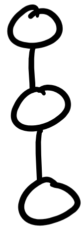
HW1 - Solution

Friday, January 16, 2026 3:36 PM

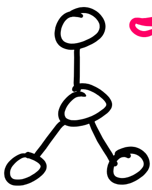
(1)



S_3

can remap two degree-1 vertices

$\Rightarrow$ <u>2 automorphisms</u>



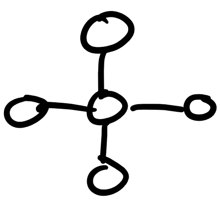
Note, any degree-1 vertex can be remapped to any other

$\rightarrow$ First vertex has 3 options

$\rightarrow$ Second has 2 independent options

$\rightarrow$ Third has 1 final option

$\Rightarrow 3 * 2 * 1 =$ <u>6 automorphisms</u>



Same logic as above

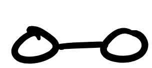
$\Rightarrow 4 * 3 * 2 * 1 =$ <u>24 automorphisms</u>

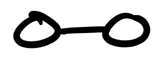
We note in general

$\rightarrow$ S_n has $n-1$ degree-1 vertices

$\rightarrow$ we map with $\begin{Bmatrix} n-1 \\ n-2 \\ \vdots \\ 1 \end{Bmatrix}$ choices in order

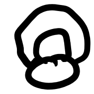
$\Rightarrow$ we have $\boxed{(n-1)! \text{ automorphisms for } S_n}$

(2) a) $n = 2 \rightarrow G_1$:  G_2: 

(6) a) $n = 2 \rightarrow G_1$:  G_2: 

→ Obviously $|E(G_2)| \neq |E(G_2)| \Rightarrow G_1 \ncong G_2$

→ Only option for $n=1$ that fits given properties is trivial graph so $n=2$ is minimum

b) $n = 1 \rightarrow G_1$:  G_2: 

→ $|E(G_2)| \neq |E(G_2)| \Rightarrow G_1 \ncong G_2$

→ $n=0$ has null graph as only option so $n=1$ is minimum

c) $n = 1 \rightarrow G_1$:  G_2: 

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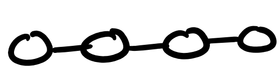
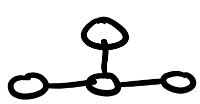
→ $n=0$ has null graph as only option so $n=1$ is minimum

d) $n = 4 \rightarrow G_1$:  G_2: 

→ degrees sequences of G_1, G_2 are not equivalents $\Rightarrow G_1 \ncong G_2$

$n=3$ → only connected acyclic graph is 

graph is o–o–o

$n=2 \to$ only option is o–o

$n=1 \to$ only option is o

$\Rightarrow$ so $n=4$ is minimum

(3) Note that we have:

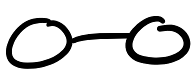
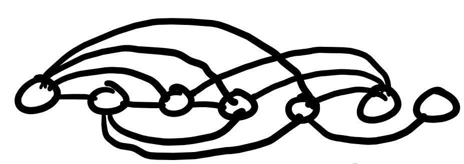
- Connected graph G
- $\forall v \in V(G): d(v) = \text{even}$

$\to G$ has an Euler Tour T

$\Rightarrow$ so we can traverse forward or backward along T to get from any u to any v via two edge-disjoint trails

$\Rightarrow$ Removing a single edge will not disconnect any u from any v due to the above, so no $e \in E(G)$ is a cut edge $\square$

(4)

S:  L:  O: degree of 7 not possible in 6-vertex graph

trivial

~~4~~ 4 4 4 4 4 3 1

trivial $\cancel{4}\ 4\ 4\ 4\ 4\ 3\ 1$

$\cancel{3}\ 3\ 3\ 3\ 3\ 1$

$2\ 2\ 2\ 3\ 1$

$\cancel{3}\ 2\ 2\ 2\ 1$

$\cancel{2}\ 1\ 1\ 1$

$1\ 1$

using H-H alg.

possible in 6-vertex graph

J: 

trivial

(5) Straight forward using pidgeon hole principle

First assume connected graph

→ we have n vertices each with a maximum degree of n-1

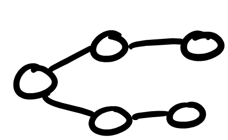
→ we are assigning n items to one of n-1 selections

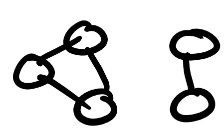
=> By PHP, at least one vertex will be assigned a degree same as some other vertex

On disconnected graph => apply logic to any component □

(6) Disprove via counter-example:

$S = \{2\ 2\ 2\ 1\ 1\}$

G_1: 

G_2: 

$G_1 \not\cong G_2$

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⑦ G is 3-regular, show:

G is bipartite $\iff$ G has decomposition into S_4 graphs

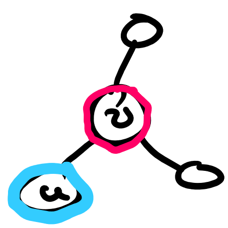
($\Rightarrow$) We can trivially construct our S_4 decomposition in X,Y-bigraph

→ $\forall v \in X$: create an S_4 by taking v and its 3 incident edges

→ Doing so will utilize all $e \in E(G)$, as each edge connects to some $v \in X$

$\Rightarrow$ Hence it is a proper decomposition into S_4 graphs

($\Leftarrow$) Consider the central vertex to each claw graph:



and outer vertices

Each v will have no other edges

→ all v form an idependent set

Note: the outer vertices u in each claw are not from the center of some

Note: the outer vertices u in each claw cannot form the center of some other claw in the decomposition

→ all u vertices must have 2 other incident edges that are also incident on 2 separate central v vertices

→ so no seperate outer vertices are adjacent, so all of u forms another independent set

=> we have all central vertices of S_4 in one indep. set and all outer vertices in a seperate indep. set

=> Thus we have a bipartition and G must be bipartite □

8 G has no odd cycles => G bipartite

We'll do strong induction on $|E(G)|$

Assume we have $P(n)$ with no odd cycles

Assume we have [illegible] with no odd cycles

Create $P(k) = P(n) - e$ by deleting some edge

→ edge deletion cannot create an odd cycle

By I.H. on $P(k)$, we have a X, Y bipartite graph

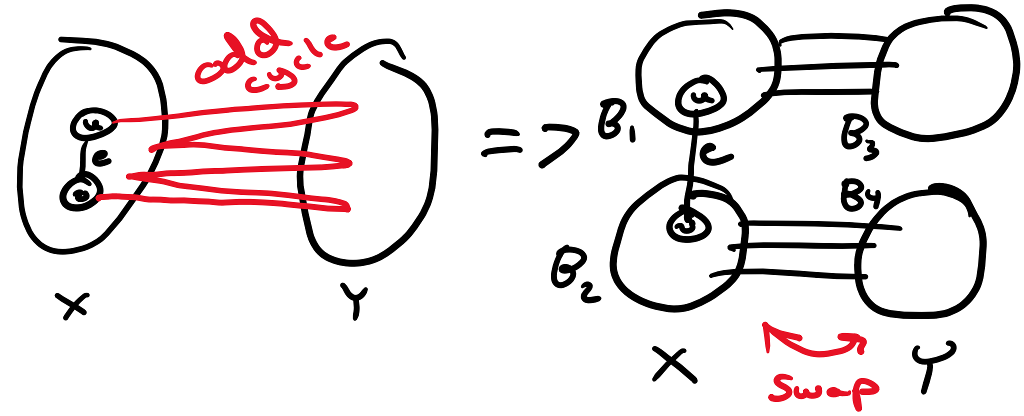
Note: $V(P(k)) = V(P(n))$
so bipartition can apply to both

We add back edge $e = (u, v)$

Case 1: $u \in X, y \in Y$ (wlog) ⇒ bipartition on $P(n)$ is still valid ✓

Case 2: $u \in X, y \in X$ (wlog)

→ This implies e must be a cut edge, as it would otherwise imply the existence of an odd cycle



→ Hence, we have the following:

$X = \{B_1, B_2\}$ with $u \in B_1$, $v \in B_2$

$Y = \{B_3, B_4\}$

$\forall e = (x,y) \in E(P(n))$: $x \in B_1$ $y \in B_3$

or $x \in B_2$ $y \in B_4$

=> We can there create a valid bipartition of $P(n)$ by swapping B_2 and B_4 s.t. $X = \{B_1 B_4\}$

$Y = \{B_2 B_3\}$ $\square$

9 On digraph D:

$\exists$ closed directed trail containing all $e \in E(D)$

$\Rightarrow \forall v \in V(D): d^+(v) = d^-(v)$

Consider any $v \in V(D)$

→ along the trail every time we reach

→ along the trail every time we reach v via an in-edge we must also exit via an out-edge

→ this holds for all vertices and accounts for all edges per our assumption

→ so in degrees and out degrees must be equal for all vertices

$$\Rightarrow \forall v \in V(D) : d^-(v) = d^+(v) \quad \square$$