Graph Theory Homework 1
Due: 30 Jan 2026 at midnight EST as a PDF on Submitty
v1.1: Last Updated January 16, 2026

1. Determine the number of automorphisms of S_3, S_4, S_5. Now, generalize this to the number of automorphisms of S_n in terms of n. (4 pts)

2. Identify the smallest n for the classes of undirected graphs defined below, such that there exists two non-isomorphic graphs G, G' within that class with $|V(G)| = |V(G')| = n$. Prove each response. (2 pts each)

   (a) Bipartite graphs.

   (b) Graphs containing a cycle.

   (c) Connected graphs.

   (d) Connected acyclic graphs.

3. Assume a simple undirected and connected graph G can be realized by the below graphic sequence. Does G have a cut edge? Prove your response. *Hint: please don't attempt to draw a realization.* (4 pts)
   $S = \{8, 8, 6, 6, 4, 6, 4, 2, 4, 2, 4, 6, 8, 10, 10, 10, 8, 4, 2, 8, 4, 6, 2, 2, 2, 2, 8, 2, 2, 2\}$

4. Which of these sequences are graphic? For the ones that are, construct a realization. (4 pts)
   $S = \{1, 1\}, L = \{4, 3, 4, 4, 4, 1, 4\}, O = \{7, 5, 2, 1, 3, 1\}, T = \{1, 1, 1, 1\}$

5. Consider simple graph G with $|V(G)| \geq 2$. Prove that $\exists u, v \in V(G) : d(u) = d(v)$. *Hint: make sure you consider whether G is connected or not.* (4 pts)

6. Prove or disprove: all possible realizations from a given graphic sequence are guaranteed to be isomorphic. (4 pts)

7. Prove the equivalence between graph classes C_1 and C_2. All graphs in both C_1 and C_2 are 3-regular. C_1 contains bipartite graphs. C_2 contains graphs that have at least one decomposition comprised entirely of S_4 subgraphs. (4 pts)

8. Using induction, prove that if G contains no odd cycles then G is bipartite. (4 pts)

9. Consider digraph D. Prove that if there exists a **closed** (v1.1) directed trail on D that contains all edges in $E(D)$, then $\forall v \in V(D) : d^-(v) = d^+(v)$. (4 pts)