HW2 - Solution

Monday, February 9, 2026 2:55 PM

① We must show each: M, S, T

T: we add a leaf each iteration, which will keep the result connected + acyclic

S: we do not halt until all $v \in V$ are added

M: Proof by **contradiction**

Assume we have a spanning tree output that is <u>not</u> maximum

define: T^* = actual MST

T = hypothetical output

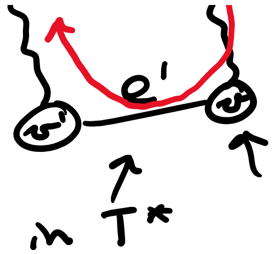
consider $e \in E(T)$, $e \notin E(T^*)$

where e is the first such edge selected by Prim's

consider $T' = T^* + e$

which creates a cycle

← Note: wlog, one u endpoint of e must be in Prim's output



be in Prim's output when e was added

Note 2: as we traverse the cycle, we will find some v in Prim's output where its neighbor v' is not in that output

→ both e and e' where available for selection, so $w(e) \le w(e')$

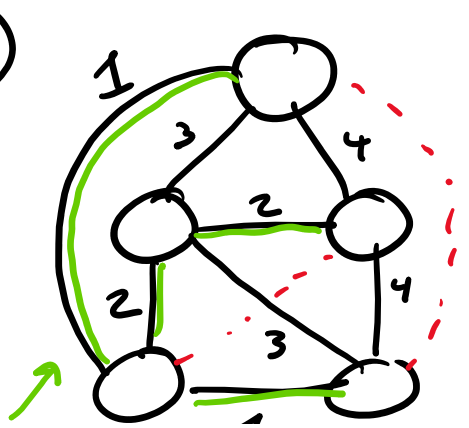
if $w(e) < w(e')$ → contradiction

as $w(T^* + e - e') < w(T^*)$

still spanning tree

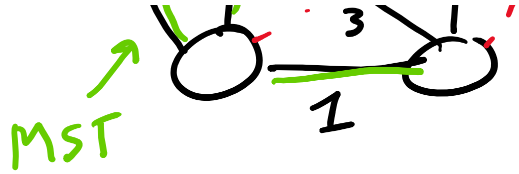
if $w(e) = w(e')$

→ repeat process above, either we encounter a contradiction or transform T^* into T □

② 

deleted

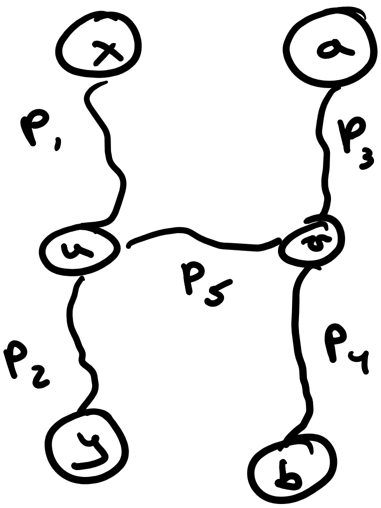
- Lowest values assigned in way to create spanning tree
- Values of 3, 4 are



– Values of 3, 4 are placed s.t. endpoints have more optimal choices

$\Rightarrow$ No other MST possible $\square$

(3) Assume we have 2 independent paths



x,y-path $= P_1 + P_2$

a,b-path $= P_3 + P_4$

$|P_1 + P_2| = |P_3 + P_4| = \ell$

$|P_5| \geq 1$

w.l.o.g. $|P_1| \geq |P_2|$

$|P_3| \geq |P_4|$

$\longrightarrow |P_1| + |P_3| + |P_5| > \ell$

contradiction

$\Rightarrow$ so $|P_5| = 0$ aka the paths must intersect $\square$

(4) PROOF BY ALGO

arbitrary tree T

Consider arbitrary tree T

repeat until $V(T) = 1$
  ℓ = leaf in T
  $T = T - \ell$

we can reconstruct T by reversing process above starting with a single vertex tree & adding leaves

⑤ Maximally acyclic => tree

tree = {simple, acyclic, connected)

acyclic => {simple, acyclic}
(obviously)

Assume max. acyclic graph is not connected

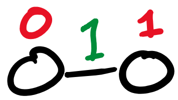
→ we can arbitrarily add an edge between vertices in any seperate components without creating a cycle

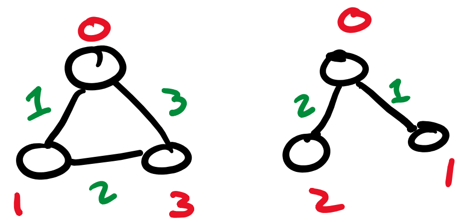
contradiction
(lot of these)

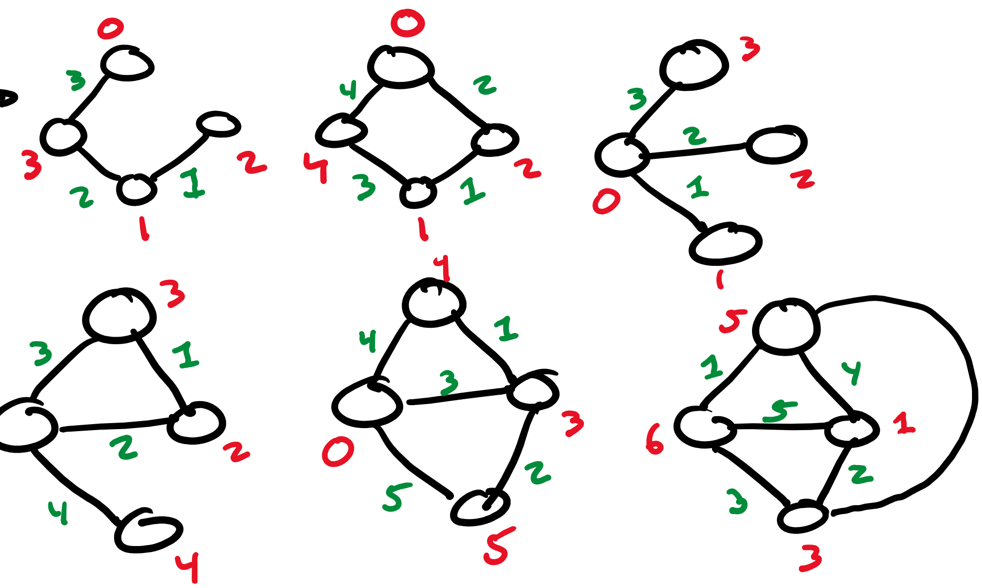
=> so a max. acyclic graph must be connected □

⑥ Proof by example

n=1 →  ← vertex => trivially graceful

n=2 →  => graceful

n=3 → 

n=4 → 

⑦ A key assumption of Djiksta:

→ when we finalize a vertex, it has a optimal value

Consider SSSP from vertex v

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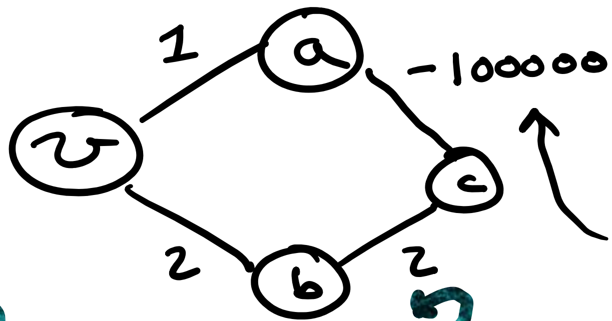

→ we finalize vertex a with a path of length 1 on 2nd iter.

However, path $\{v, b, c, a\}$ has length -99996

counter-example

$\Rightarrow$ a negative value can result in a contradiction of that above key assumption $\square$

(8) G has one cycle $\Leftrightarrow |V(G)| = |E(G)|$

($\Rightarrow$) Consider spanning tree T of G

$\rightarrow |E(T)| = \underbrace{|V(T)| - 1}$, $|V(T)| = |V(G)|$

$|V(T)| = \underline{|E(G)|}$

we can create T by simply deleting a single edge from G

→ we must do so by selecting some edge on the single cycle

As $G = T + e \Rightarrow |E(G)| = \underbrace{E(T)} + 1$

$= \underline{|V(T)|} - 1 + 1$

$= |V(G)| - 1 + 1$

$= |V(G)|$ ✓


(strong)

($\Leftarrow$) Induction on $|E(G)|$

Basis: all $G \in C_i \rightarrow |V(G)| = |E(G)|$
(all cycle graphs)

$P(n)$: some connected graph with
$|V(P(n))| = |E(P(n))|$

$P(k) = P(n) \cdot e \rightarrow |V(P(k))| = |E(P(k))|$
$= |E(P(n))| - 1$ ✓

Case 1: e is not on a cycle

$\rightarrow$ I.H. on $P(k)$ to assume single C_i

$\Rightarrow$ uncontraction of e does not create or delete cycles on $P(n)$

Case 2: e is on a cycle

$\rightarrow$ I.H. on $P(k)$ to assume single C_i

$\Rightarrow$ uncontracting e simply results in C_{i+1} on $P(n)$, but does otherwise not create or delete other cycles $\square$