HW3 - Solution

Thursday, February 19, 2026 3:08 PM

1. Strong induction on $\alpha'(G) - |M|$

(Note: we'll prove a stronger result on general graphs)

Basis $P(0)$: we have a max match trivially $\rightarrow$ any unsaturated v will not have an M-aug. path per Berge and is not matched on our assumed max match

$P(n)$: we have $n = \alpha'(G) - |M|$ for some match M

we assume we have some v w/no M-aug path

$P(k)$: we create $M_0 = M \Delta P$ for some M-aug path P $\leftarrow$ Berge says P must exist

$(k = n-1)$

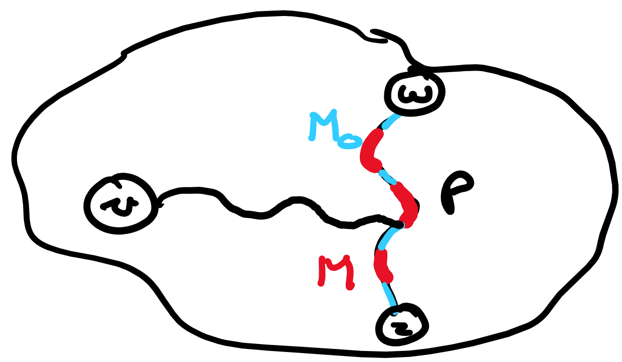
To use our I.H. we must show:

a) v is unsaturated on $P(k)$

$\rightarrow$ trivially, v is independent of P

b) v does not have any M_0-aug. paths on $P(k)$

→ consider the below configuration



we note the only difference between M and M_0 is on P

If v has M_0-aug path from v to z (wlog), this implies v has an M-aug path from v to w on P(n)

contradiction

→ so no M_0-aug path exists

I.H. → v is unsaturated on max match

P(k) → P(n)

we undo our construction and note trivially v is unaffected

⇒ v is unsaturated on a max match □

(2) We know $M \Delta M'$ is always alternating paths or cycles for matches M, M'

$\rightarrow$ cycles always even # $e \in M, e' \in M'$

$\Rightarrow$ as $|M| < |M'|$, we must have at least one alternating path with more edges in $|M'|$ $\square$

(3) Consider any possible $S \subseteq V(G)$ and a resulting $G-S$

consider odd component H

$$\sum_{v \in V(H)} d(v) = 3|V(H)|$$

$$|(S,H) \text{ cut}| = c$$

$$\underbrace{2|E(H)|}_{\text{edges in } H} + \underbrace{c}_{\text{edges } S \leftrightarrow H} = \underbrace{3|V(H)|}_{\text{degree sum in } H}$$

Note: $c \neq 1$ as we have no cut edges

Note 2: c must be odd by above

parity

$\Rightarrow c \geq 3$

$\to c \geq 3$

$$\begin{cases} 3\, o(G-S) \geq \text{total cut edges} \\ 3|S| = \text{degree sum in } S \end{cases}$$

$\to$ total cut $\leq$ S degree sum

$$\Rightarrow 3\, o(G-S) \leq 3|S|$$

$$\boxed{o(G-S) \leq |S|}$$

aka Tutte's holds $\square$

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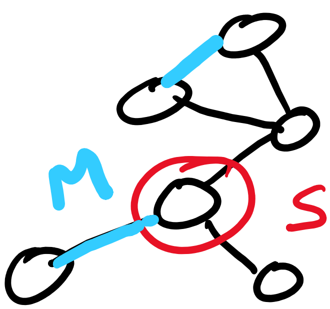

G

$o(G-S) = 3 > |S| = 1$ ✗

Tutte's fails

$|M| = 2$

5 Consider the marriage problem and some $x \in$ Men, $y \in$ Women, where (x, y) is a hypothetical unstable pair output by ALGO.

First, we note that all M+W have a partner when algo stops

→ only way for some woman (and therefore man) to be unmatched is if man never proposed to woman, but we know all men can eventually propose to all women

We have a P.M., can unstable (x,y) pair exist?

→ one of the following must exist

Case 1: x never proposed to y

→ x must prefer his current partner more than y

Case 2: y rejected x

→ y only rejects for a higher preference partner

=> both cases contradict the assumption of unstable pair □

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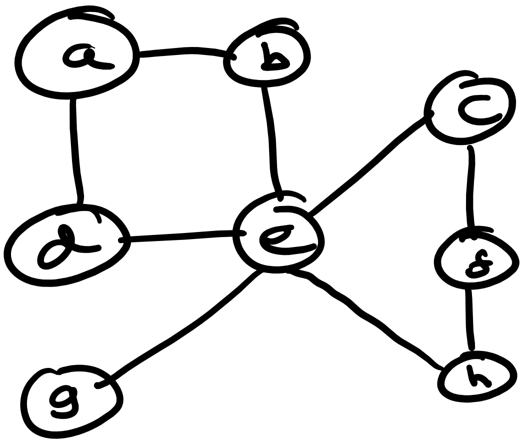

a) $\{a, e, h\}$ is min cover and dom. set

b) $|C| = |M'|$ as G is bipartite + König-Egerváry

c) $\overline{C} = \{b, d, g, c, h\}$

→ we note pairwise no 2 verts in C are neighbors ✓