HW5 - Solution

Wednesday, April 1, 2026 4:05 PM

① Consider some $k = \chi(G)$ - coloring of G

- order colors $C_1 C_2 \ldots C_k$
- this is our order for greedy coloring, by considering verts with each C_i

↳ a vertex in color group C_i can only depend on at most $(i-1)$ colors when processed by G.C.

⟹ no vertex will get color higher than k □

② Our worst-case for G.C.

↳ $\chi(G) \leq \Delta(G) + 1$

Many counter-examples exist (graphs that can never hit that bound with any ordering)

E.g. trees, C_4 w/ a chord

Most EXXXXTREEEME example:

$S_n \to$ 2-colorable with any order

but $\Delta(G)+1 = n >>>>>> \chi(G) = 2$

as $n \to \infty$

(See: Grundy Number)

(3) We'll construct a coloring where colors are pairs

$\to \forall v \in V(G): c(v) = (c_1(v), c_2(v))$

where $c_1(v)$ is color in $\chi(G_1)$-coloring of graph G_1

and same for $c_2(v)$ and G_2

I.e., each vertex gets a color pair

Note 1: at most $\chi(G_1) * \chi(G_2)$ pairs

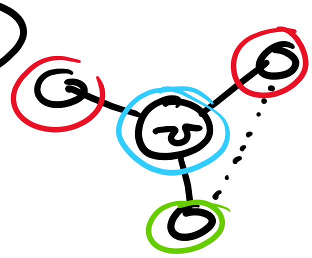
Note 2: $\forall e \in E(H)$, e is from G_1 or G_2

$\hookrightarrow$ so no $(u,v) = e$ can have the same color pairs at each endpoint $\square$ (u, v can have same color in at most one of G_1, G_2)

(4) As $\dfrac{|V(G)|}{\alpha(G)}$ is the minimum number of independent sets, $\chi(G)$ is

of independent sets, $\chi(G)$ is necessarily bounded below by it

Recall: minimum coloring is equivalent to finding a min # of I.S.s $\square$

⑤ 

→ in claw-free graph, any v can have at most 2 neighbors in any independent set

→ Inducing a graph on vertices in 2 color class gives a max degree of 2

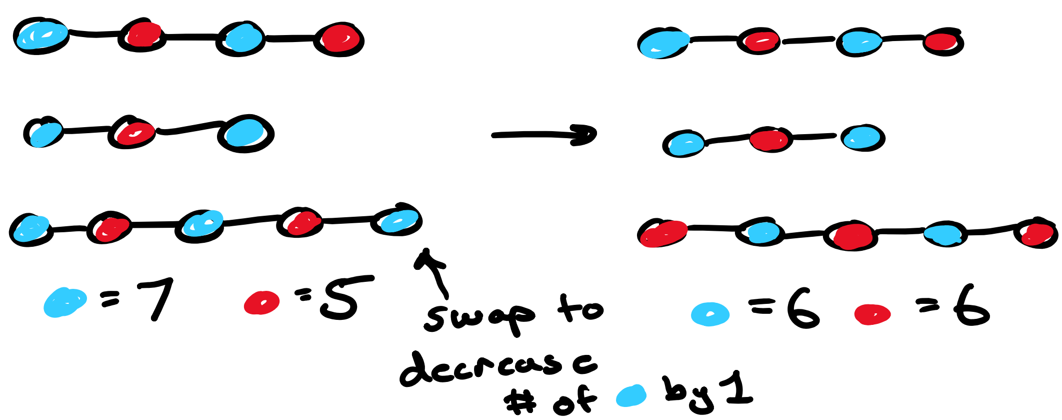
→ we have paths and cycles

→ must be even to alternate colors $\square$

⑥ Consider the above result:

→ any graph induced on two color classes will be paths and cycles

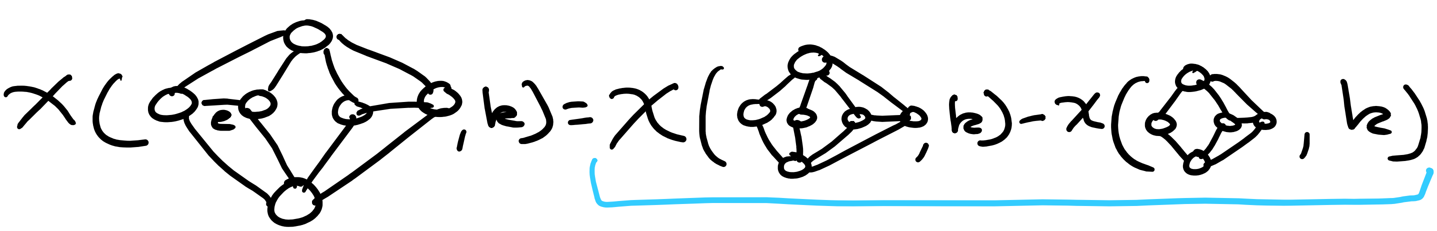
For any arbitrary (C_1, C_2) color pair, we can recolor vertices on paths such that we reach equality +/- 1

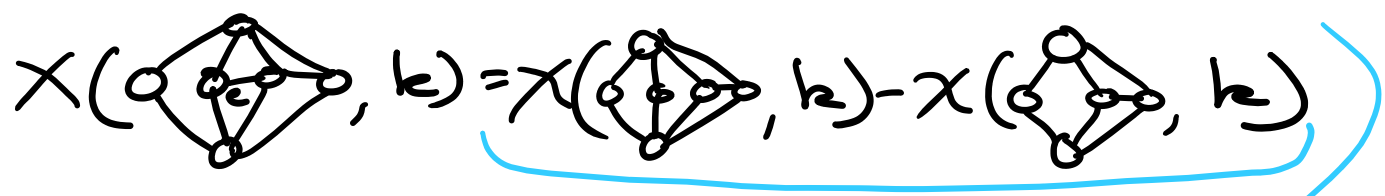


→ swapping colors on a path won't impact the global proper coloring (we'll see this later w/ Kempe chains)

⇒ doing this among all color pairs will produce the desired result □

(7) We'll use one step of our recurrence to show this: $\chi(G,k) = \chi(G-e,k) - \chi(G\cdot e,k)$

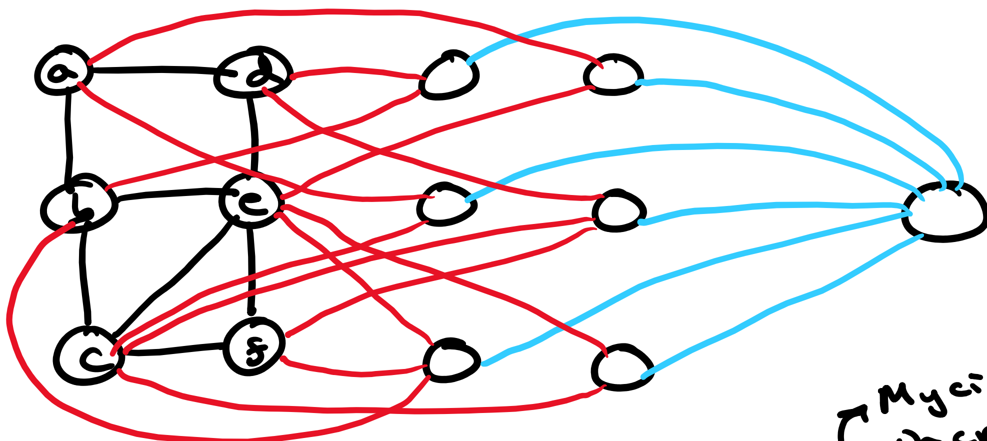




these are equal □

⑧

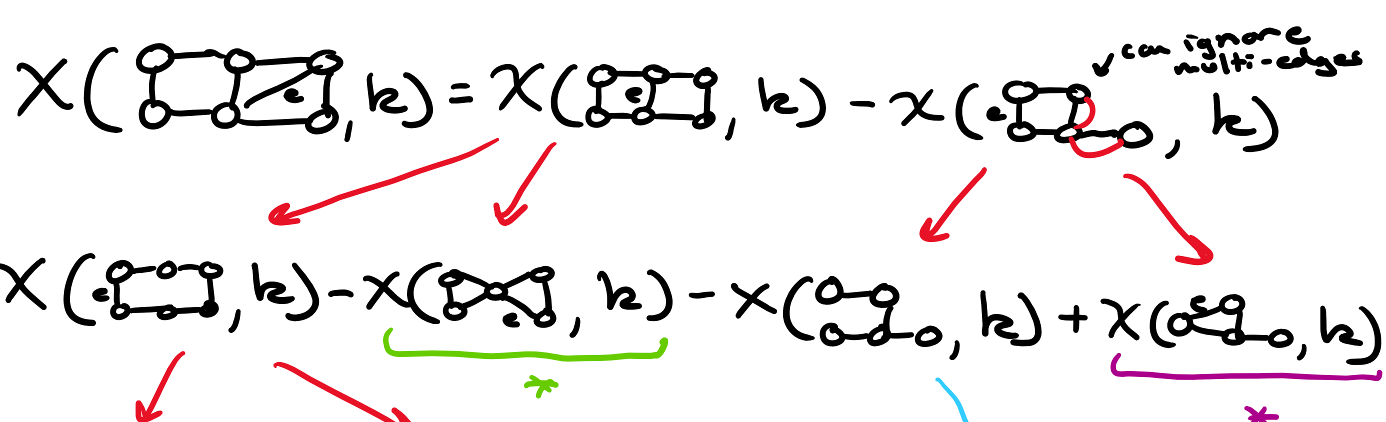
a)

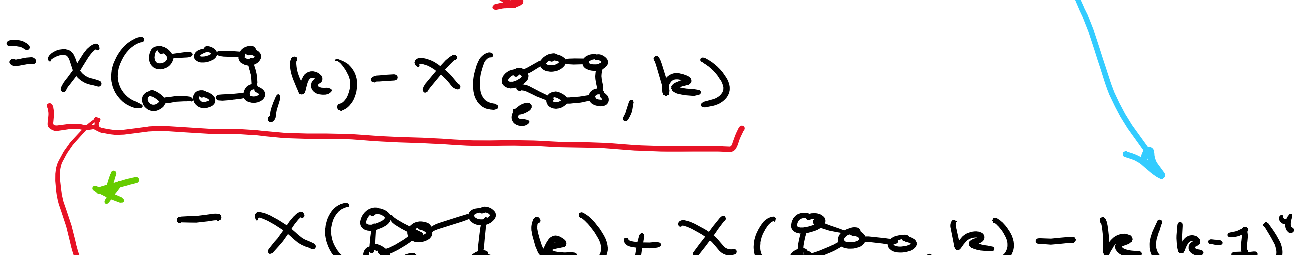


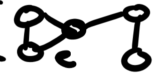
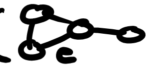
$\omega(G) = 3,\ \omega(G') = 3$ } Mycielski will not increase the clique number

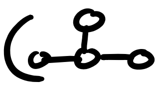
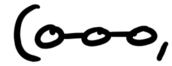
$\chi(G) = 3,\ \chi(G') = 4$ } we know the construction will always increase the chromatic number by 1 □

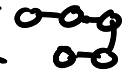
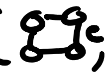
b)

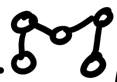
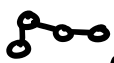
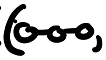




$-\chi($$, k) + \chi($$, k) - k(k-1)^4$

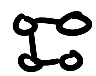
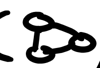
$+ \chi($$, k) - \chi($$, k)$

$= k(k-1)^5 - \chi($$, k) + \chi($$, k)$

$-\chi($$, k) + \chi($$, k) + \chi($$, k) - \chi($$, k)$

$- k(k-1)^4$

$+ k(k-1)^3 - k(k-1)^2$

$= k(k-1)^5 - k(k-1)^4 + \chi($$, k) - \chi($$, k)$

$- k(k-1)^4 + k(k-1)^3 + k(k-1)^3 - k(k-1)^2 - k(k-1)^4$

$+ k(k-1)^3 - k(k-1)^2$

$$= k(k-1)^5 - k(k-1)^4 + k(k-1)^3 - k(k-1)(k-2)$$
$$- k(k-1)^4 + k(k-1)^3 + k(k-1)^3 - k(k-1)^2$$
$$- k(k-1)^4 + k(k-1)^3 - k(k-1)^2$$

$$= k(k-1)^5 - 3k(k-1)^4 + 4k(k-1)^3 - 2k(k-1)^2$$
$$- k(k-1)(k-2)$$

(sorry, this was pretty stupid)

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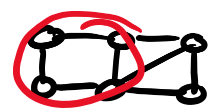
$$\chi(G,1) = 0 - 0 + 0 - 0 - 0 = 0$$

$$\chi(G,2) = 2 - 6 + 8 - 4 - 0 = 0$$

$$\chi(G,3) = 96 - 144 + 96 - 24 - 6 = 18$$ ✓

$$\boxed{\chi(G) = 3}$$

c) G is not chordal 

→ In class we proved the equivalence

G chordal ⇔ G has SEO

contrapositive

G not chordal ⇔ $\underbrace{\text{G has no SEO}}$

↳ no SEO ✓

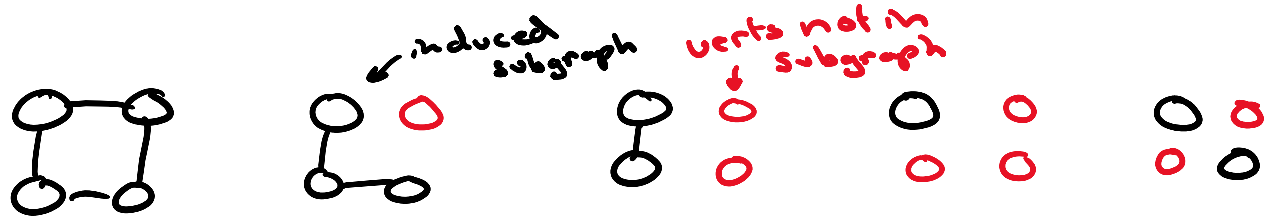
In class, we noted

chordal graphs ⇒ perfect

Note: this is <u>not</u> an equivalence

I.e., a perfect graph can have a chordless cycle

→ C_4 is an obvious example



$\chi(C_4)=2$	$\chi(P_3)=2$	$\chi(P_2)=2$	$\chi(P_1)=1$	$\chi=1$
$\omega(C_4)=2$	$\omega(P_3)=2$	$\omega(P_2)=2$	$\omega(P_1)=2$	$\omega=1$

$\Rightarrow$ so it having no SEO says <u>nothing</u> about it being perfect □