HW6 - Solutions

Thursday, April 9, 2026 5:19 PM

① We can consider connected G and a disconnected G in the same way

Prove: any face can reach the outer face by crossing some edges

e.g. any $v \in V(G^*)$ can reach the outer face vertex v_f

$\rightarrow \forall u, v \in V(G^*) : \exists\, u,v$-path

$\hookrightarrow \{u, v_f\}$ then $\{v_f, v\}$

PROOF BY ALGO?

- Consider an infinite line drawn from our starting face $\rightarrow v_0 \in V(G^*)$ to the outer face

- We cross some number of edges

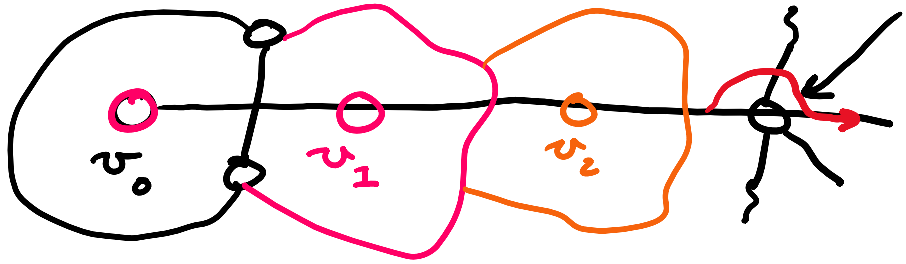
- If we cross an edge separating faces, we traverse from $v_i \rightarrow v_{i+1}$ in G^* on edge (v_i, v_{i-1})

$\rightarrow$ we eventually reach v_f on edges of G^*

... ly reach v_s

on edges of G^*

- If we exactly reach a vertex w in G

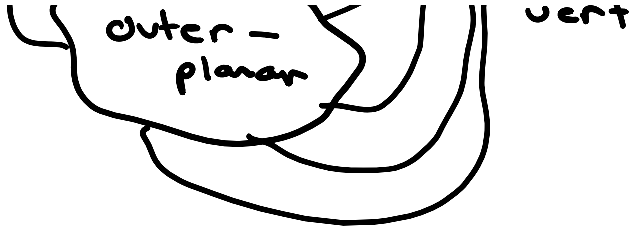


→ we re-route around w by crossing its incident edges in a clockwise manor

→ the same logic as above applies

$\Rightarrow$ as all $v_0 \in V(G^*)$ can reach v_s, G^* is connected □

② Recall our construction that showed we can trivially add a new vertex and connect it to all vertices in an outerplanar graph



→ the new vertex will increase the chromatic number exactly by 1 and result is planar

→ We know all $\chi(G) \leq 4$ for a planar G

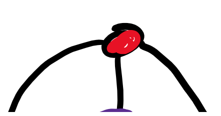
$\Rightarrow$ So any outerplanar G has $\chi(G) \leq 3$ $\square$

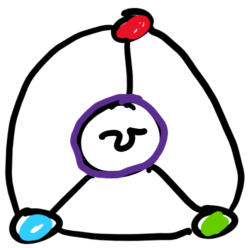
(3) $\chi(G) \leq 3 \iff \forall v \in V(G): d(v) = \text{even}$

($\Rightarrow$) contrapositive

$$\exists v : d(v) = \text{odd} \Rightarrow \chi(G) \geq 4$$

consider any odd v in a triangulation configuration ($N(v)$ in a ring)





← Note the ring around v is an odd cycle

→ requires 3 colors

→ so v needs 4th color

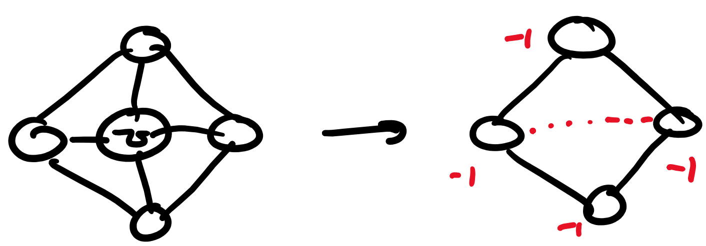
$\Rightarrow \chi(G) \geq 4$ ✓

($\Leftarrow$) We can do induction similar to our 4-color proof

Note: we know $\exists v : d(v) \leq 5$

$\rightarrow \exists v : d(v) = 4$

Note 2: we can't just delete v for our construction

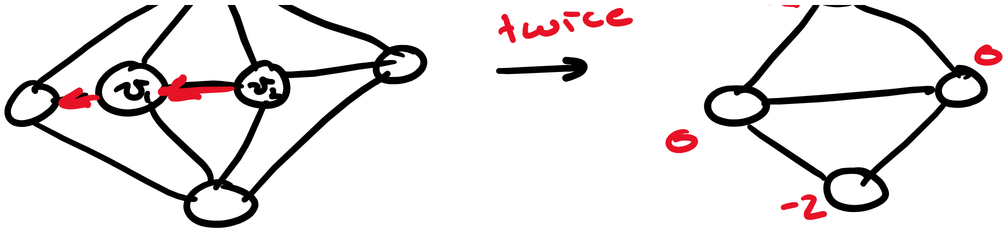


← not a triangulation, ring degrees also odd ✗

What aboot 2 adjacent v?

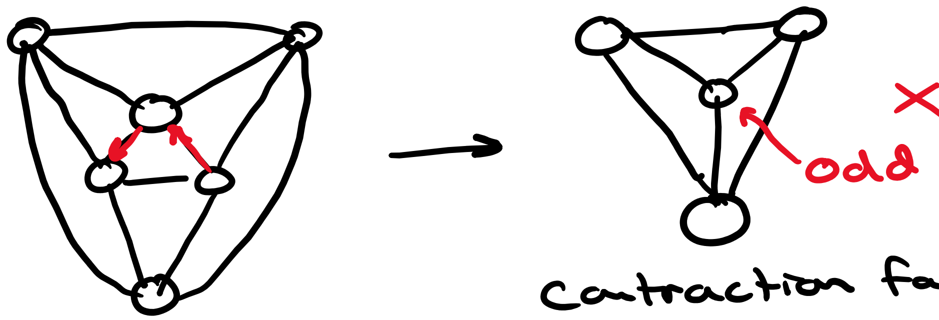
AND edge contraction

Now this is pod

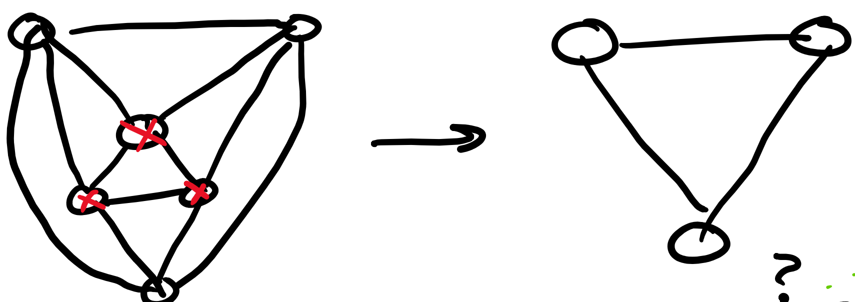


is pod racing ✓

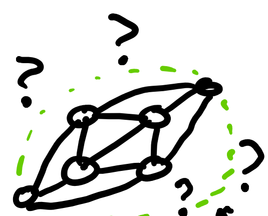
OR 3 adjacent v



contraction fails



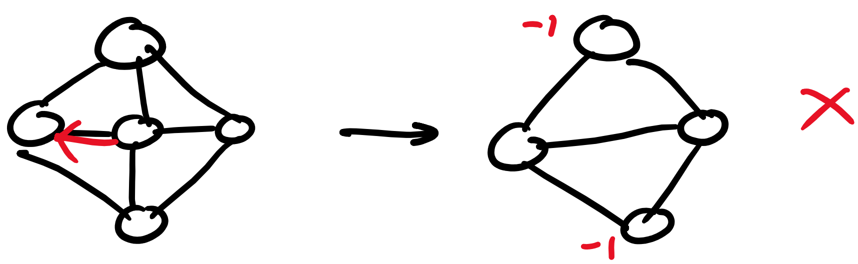
deletion works ✓



edge here is same as 3 case

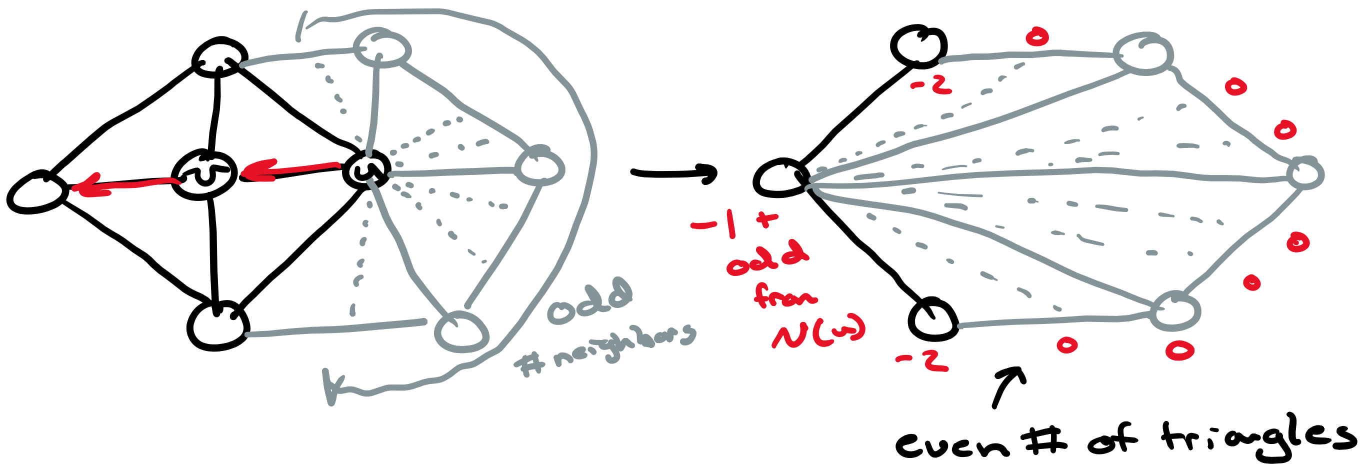
Note 3: 4+ v can't be a possible/unique configuration

Note 4: single v edge contraction won't work

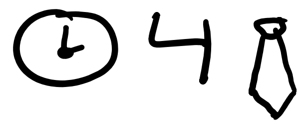


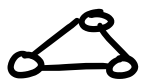
what if only a single v?

$\exists u \in N(v) : d(u) \geq 6$

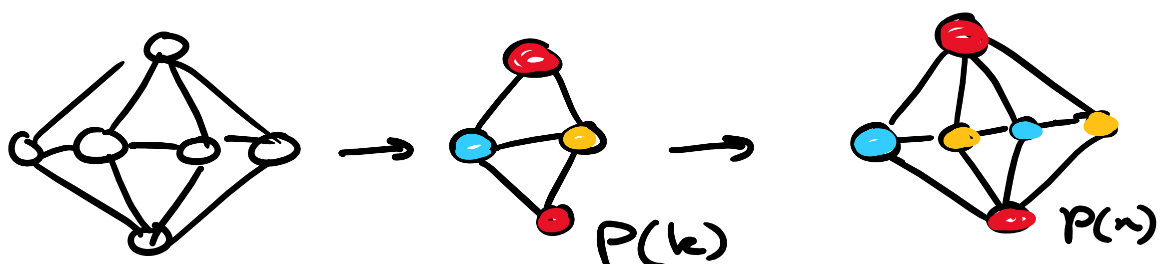


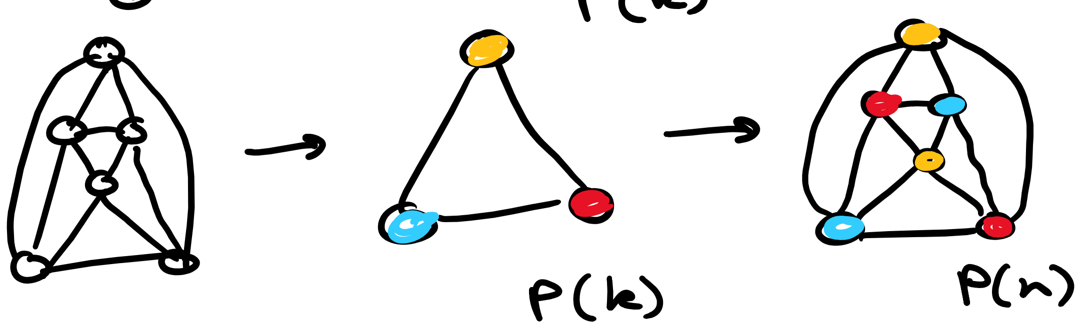
Time for biznitz

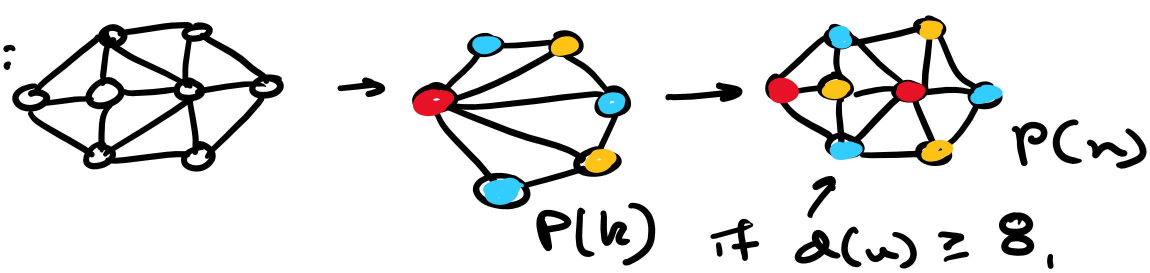


Basis:  → trivial

$P(k)$ = one of several constructions

Case 1: 

Case 2: 

Case 3: 

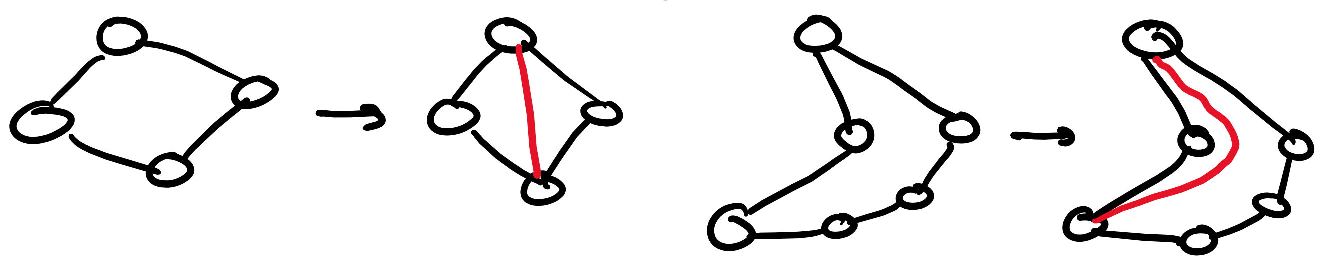
if $d(w) \geq 8$,
colors alternate an
even # more times

colors alternate an
even # more times
→ v, u colored same

=> all cases covered, and we
have a 3-coloring for all ☑

④ PROOF BY ALGO.

Note: any chordless cycle as a
face in embedding can trivially
get a chord drawn while
remaining planar



→ can follow other edges arbitrarily
closely to connect vert pair

Our (ultra) complex algo:

While a chordless C exists:

connect 2 non-connected vertices with a new chord

$\Rightarrow$ we end up with a planar graph where all faces are triangles $\square$

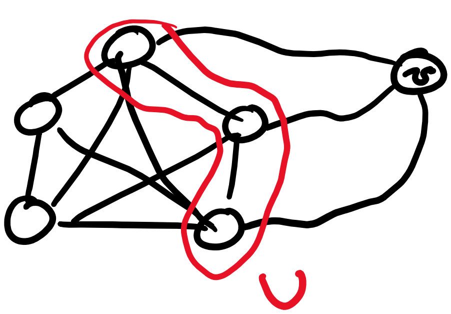
5. contrapositive

Not a triangulation $\Rightarrow$ not max planar

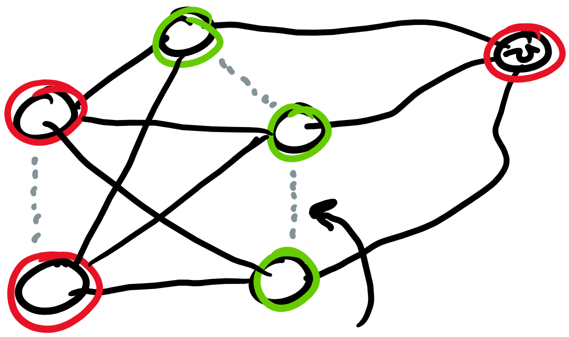
$\exists$ some face of at least length 4

$\rightarrow$ using result from 4, we can trivially add a chord to that face

$\Rightarrow$ the graph isn't maximally planar $\square$

6. 

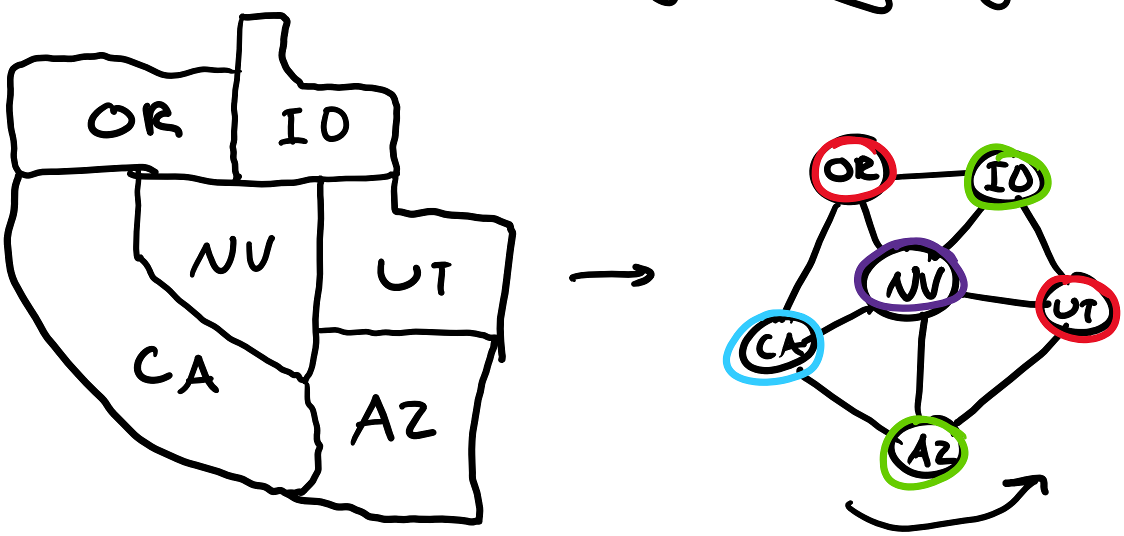
$\leftarrow$ there exists at least one vertex v with a v, U - fan to some $U \in K_5$-sub



$\Rightarrow$ we can easily observe a $K_{3,3}$ subdivision $\square$

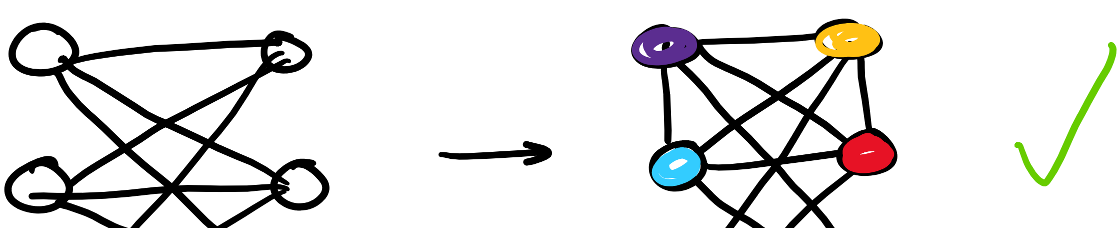
Note: v can be on a K_5-sub edge

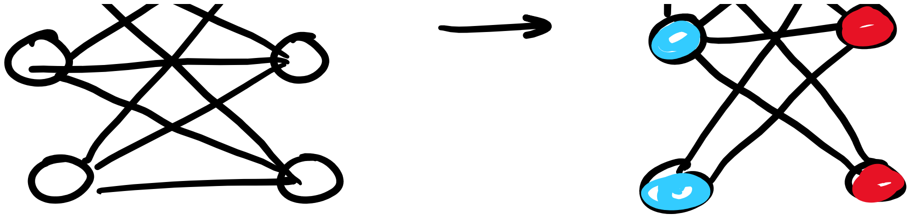
⑦ Any map with a region bordered by an odd # of neighboring regions works



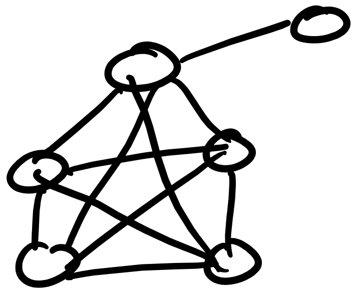
need 3 colors for odd ring +1 for Nevada $\square$

⑧ a) $K_{3,3}$ with some extra edges to create K_4 is nonplanar 4-colorable





b) K_5 + an extra v and edge is nonplanar



$|V| = 6$

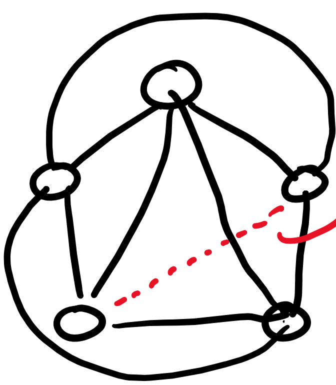
$|E| = 11$

$e \le 3n - 6$

$11 \le 12$

c) Literally any minimal nonplanar graph is a counter-example

e.g., K_5



edge is automorphically equivalent to any other