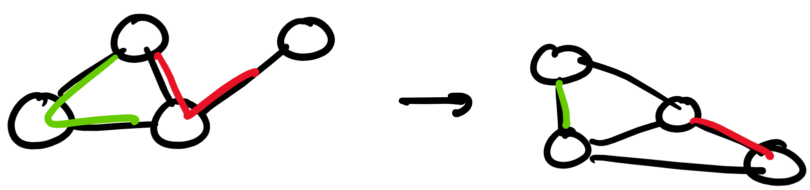
HW7 - Solution

Tuesday, April 21, 2026 1:31 PM

① Two approaches:

1: Note: a 'wedge' (i.e, P_3) in G equates to to an edge in L(G)



AND: disjoint wedges become disjoint edges in L(G)

SO: all we need to show is a decomposition of wedges always exists when $|E(G)|$ = even

=> trivial via strong induction □

2: Using Tutte's

On G, cutting k edges results in at most k+1 components

1/0 parity 1/0 1/0 1/0 → At most k of these components can have an odd number of edges

Note: For cut $|S| = k$

$G - S$ (odd) $\rightarrow$ odd edges across $k+1$ (even) components, at least one comp even

$G - S$ (even) $\rightarrow$ even edges across $k+1$ (odd) components, at least one comp even

A set of k edges on G is a set of k vertices on $L(G)$

$\rightarrow$ so our cut edges are cut vertices, and the parity of components on $L(G)$ would be the same as above

$\Rightarrow$ so $L(G)$ satisfies Tutte's for any choice of S $\square$

2. What was discussed in class via ALGORITHM

If in G_{XY}: $|X| < |Y|$ then add
(wlog)
vertices to X s.t. $|X| = |Y|$

As degree sums are equal in X, Y
if $\exists u \in X : d(u) < k$
then $\exists v \in Y : d(v) < k$
$\rightarrow$ add edge (u, v)
$\rightarrow$ iterate until all $u \in X, v \in Y$
have $d(u) = d(v) = k$

$\Rightarrow$ we have a k-regular H
where $G \subseteq H$ $\square$

3 ($\Rightarrow$) Each $e \in E(G)$ gives us
exactly one $v \in V(L(G))$

$\rightarrow$ as $G \cong L(G) \Rightarrow G \cong L(L(G))$
we have $|V(G)| = |E(G)|$
$2|E(G)| = \sum d(v) = 2|V(G)|$

$\hookrightarrow$ so average degree is 2

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Each $v \in V(G)$ contributes exactly $\binom{d(v)}{2}$ edges to $E(L(G))$

$\rightarrow$ so we also must have

$$\sum \binom{d(v)}{2} = |V(G)| = |E(G)|$$
$$= |E(L(G))|$$

However: if we have some $d(v) > 2$, then we require some $d(u) < 2$ to maintain equality

$\rightarrow$ The only way we have $d(u) = 1$ in $L(G)$ is if u is along a path

BUT: a path in $G \rightarrow$ a shorter path in $L(G)$

SO: all $d(v)$ must equal 2

$\Rightarrow G$ is 2-regular $\square$

($\Leftarrow$) Relatively trivial

$\rightarrow$ 2-regular graphs are comprised of cycle components of cycle components

$\Rightarrow$ we've observed for any C_k that $C_k \cong L(C_k)$ $\square$

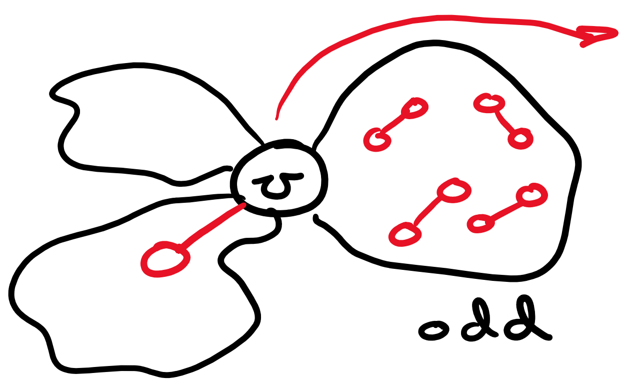
④ If $\Delta(G) = \chi'(G)$, each vertex has exactly $k = \Delta(G)$ colors incident on them

$\rightarrow$ each color forms a perfect match

$\rightarrow$ $|V(G)| =$ even

So $G - v$ has at least one odd component
(cutvert)

-1/0 parity -1/0 -1/0 -1/0



There is at least one color incident on v that is not on $N(v)$ in the odd component

$\rightarrow$ That color must therefore have a P.M. on that odd must therefore have a P.M. on that odd component

Contradiction

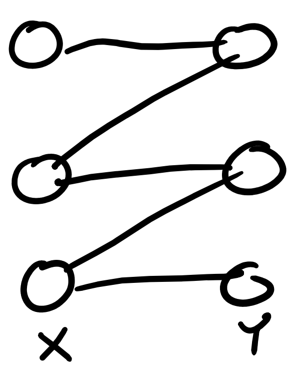
$\Rightarrow$ so we cannot have

$$\Delta(G) = \chi'(G) \quad \square$$

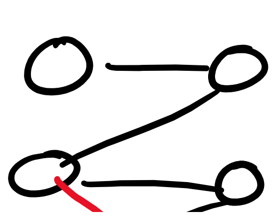
5 a) In class, we saw

G is Ham. $\Leftrightarrow$ $C(G)$ is Ham.

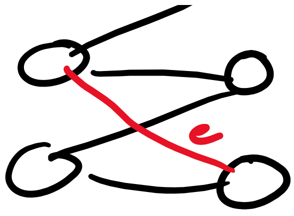
As a clique is Ham $\Rightarrow$ G is Ham. $\square$

b) 

Clearly $|X| = |Y|$ and no cycle exists

c) 

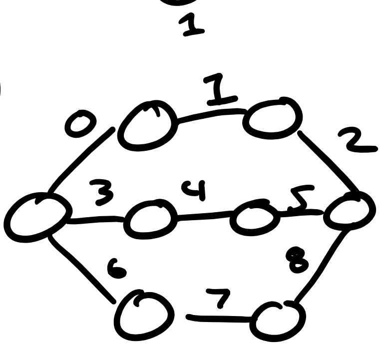
G is only Ham. if $e = (u, v)$ attaches



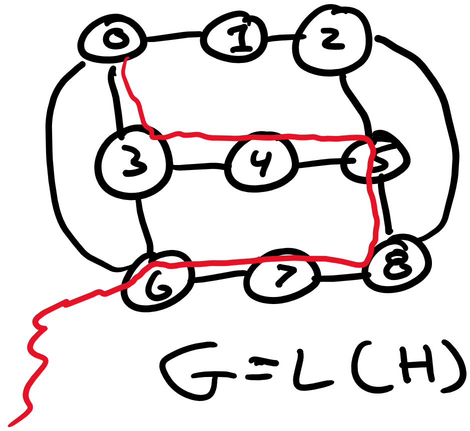
G'

$e = (u, v)$ attaches to u, v – H.P. ×

d)



H



$G = L(H)$

→ We observe there is no possible cycle containing $\{1, 4, 7\}$ in G, so no H.C. is guaranteed ×

⑥ A triangle exist between some u, v, w when all edges $(u, v), (u, w), (v, w)$ exist, each with probability p

→ we have $\binom{n}{3}$ such selections of 3 vertices of 3 vertices

→ Each triangle has a probability of $p * p * p = p^3$

$\Rightarrow$ #triangles $= \binom{n}{3} p^3$ $\square$

7 Assume suitably large n (and p s.t. $G(n,p)$ is connected)

We know $|N(v)| \approx np$

1-hop neighbors: np

2-hop neighbors: $(np)^2$

3-hop neighbors: $(np)^3$

From class: diameter is $\sim \ln(n)$

We take the averge distances:

$$\Rightarrow \frac{1}{n} \sum_{i=1}^{\lceil \ln(n) \rceil} i \underbrace{(np)^i}_{\text{verts at that distance}}$$

distance

$\bar{z} = 1$

avg. over all verts

distance

verts at that distance

sum of distances

Note: this will likely just be $\approx \ln(n)$