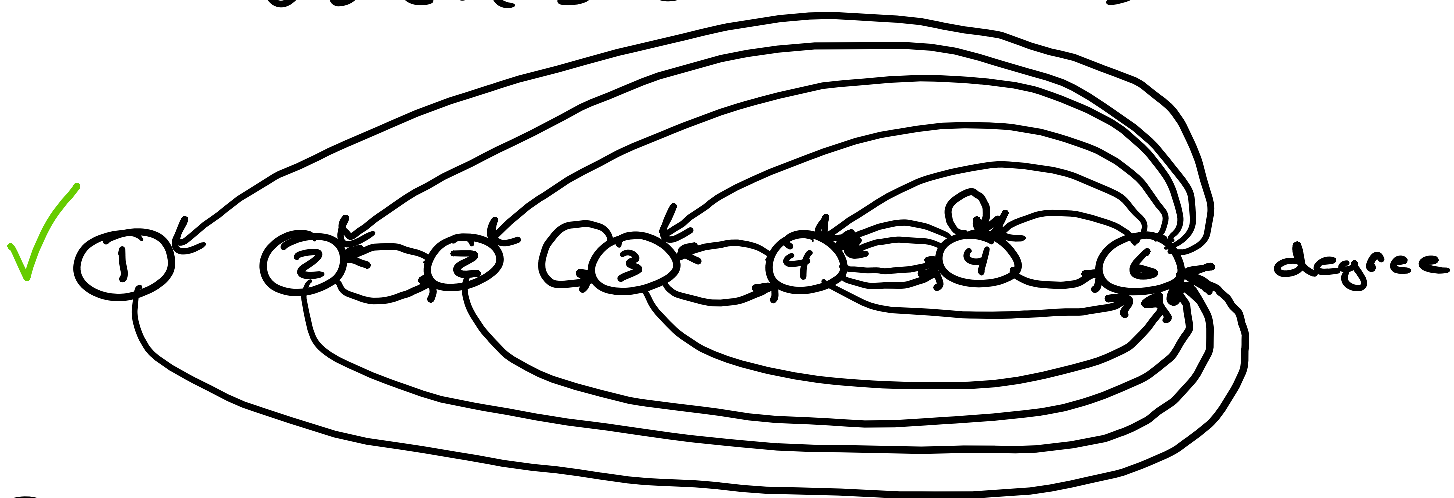
Practice Midterm - Solution

Saturday, February 21, 2026 10:59 AM

1. we need connected G where

$$\forall v \in V(G) : d^+(v) = d^-(v)$$



2. a) we know $2^{\frac{n(n-1)}{2}}$ possible undirected graphs

$\rightarrow$ 2^n possible loop configurations

$\Rightarrow$ $\boxed{2^{\frac{n(n-1)}{2}} 2^n}$ loopy graphs

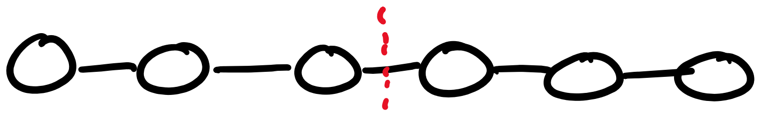
b) Instead of 1/0 edge exists/doesn't exist (in adj. matrix)

$\rightarrow$ we have 2/1/0

for 2 loops/multi-edges

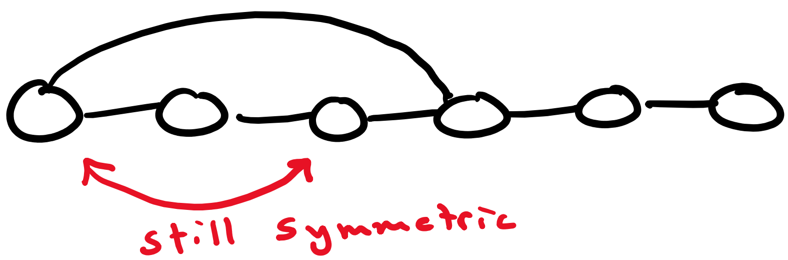
$\Rightarrow$ $\boxed{3^{\frac{n(n-1)}{2}} 3^n}$ configurations

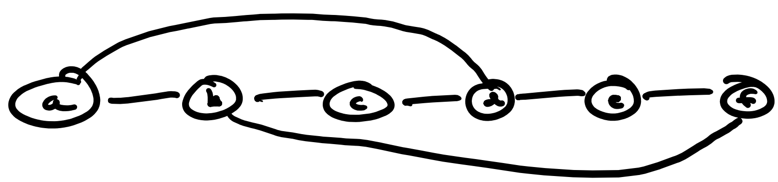
③



P_6 has 2 automorphisms around a mirror axis

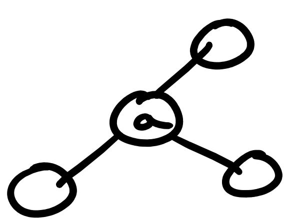
→ break that symmetry



=> 

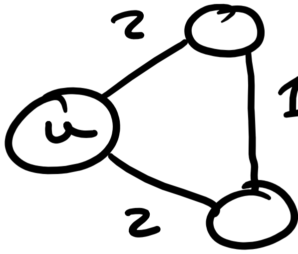
✓

④ Proof by counter-example

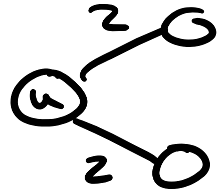


→ EZ via Tutte's with $\{a\} = S$, no P.M.

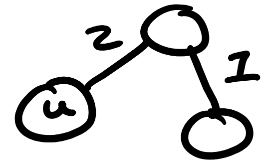
⑤ Simple to show: SSSP tree $\neq$ MST



G



SSSP from u



MST

6 $(\Rightarrow)$ Trivially, we require a graph to have an even number of edges to have a P.M.

$\rightarrow |V(G)|$ must be odd for $|V(G-v)|$ to be even ✓

Define $G' = G - v$

Per Tutte: $\forall S' \subseteq V(G') : o(G' - S') \leq |S'|$

Define $S = S' + v$

$$G - S = G' - S'$$

$$o(G-S) = o(G'-S')$$

$$|S'| = |S| - 1$$

$$\rightarrow o(G-S) = o(G'-S') \leq |S'| = |S| - 1$$

$o(G-S) \leq |S| - 1$ ✓

$(\Leftarrow)$ Using $G' = G - v$, $S = S' + v$, $S' \subseteq V(G')$

Note: $|S|$ and $o(G-S)$ must have differing parity (sum of verts in odd comps plus $|S|$ must be odd)

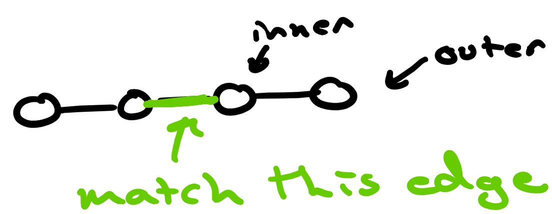
$\to o(G-S) \leq |S| - 1$ ← in order for assumption to hold given the above

$o(G'-S') = o(G-v-S+v)$

$= o(G-S) \leq |S| - 1 = |S'|$

→ Tutte's holds for G', so it has a P.M.

⑦ $(\Leftarrow)$ Consider P_4:



We note "inner" vertices have degree of 2 within a P_4, so must be an "outer" vertex in one other P_4 in the decomposition as $d(v) = 3$

→ This applies to all $v \in V(G)$

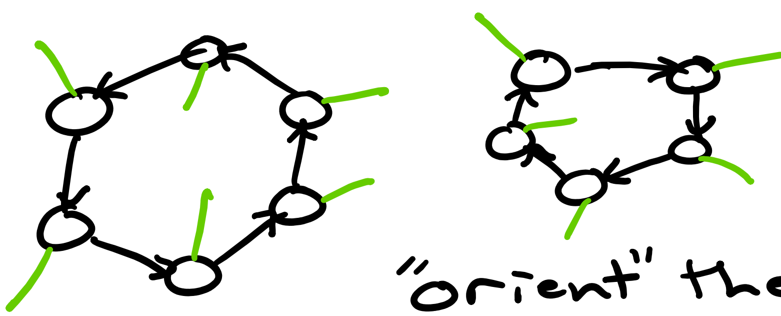
→ Matching middle edge in all P_4

→ Matching middle edge in all P_4 gives us a proper P.M. ✓

($\Rightarrow$) Consider 3-regular G with a P.M.

– delete all matched edges → G'

→ we have a 2-regular G', which is cycle subgraphs

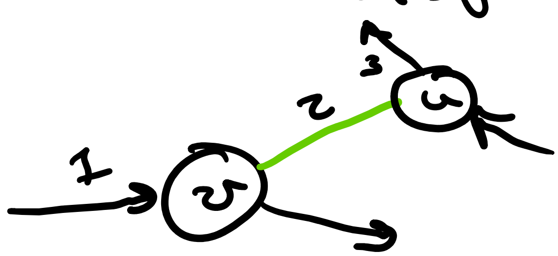


← deleted $e \in M$

"orient" the edges in one direction around each cycle in G'

make directed

Construct our disjoint P_4's



1. Take incoming edge to v
2. deleted matched edge
3. outgoing edge from v's

5. outgoing edge from v's matched partner

→ each edge used exactly once and assigned to same P_4

→ we have a proper P_4 decomp. □

Exercise 4 reader: use induction instead


strong

⑧ Induction on $|E(G)|$

Basis: all C_n graphs

→ trivially, $\forall v \in C_n : d(v) = 2$ and can be decomposed into a cycle

$P(n)$: We have even-degree graph

Note: it must have some cycle C as shown in class

$P(k) = P(n) - E(C)$
edges of cycle

% % parity

edges of cycle 1/0

→ as even − even = even,
or = 0
P(k) has all even degrees or
verts of degree 0 ← not relevant for a decomposition

parity 1/0 1/0

I.H.: P(k) decomposed into cycles

P(k) → P(n)

we add back E(c) and include
the cycle in P(k)'s decomp to
get a proper decomp on P(n) □

(9) Every maximum independent set I
is trivially a dominating set D

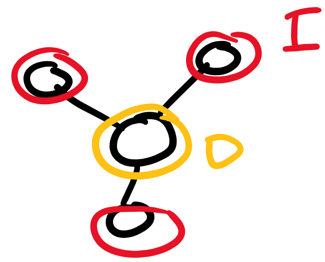
→ if some vertex is not in or adjacent
to I, it could be added to I

→ so a minimum D cannot be larger
than a maximum I

→ $|D| \leq |I|$

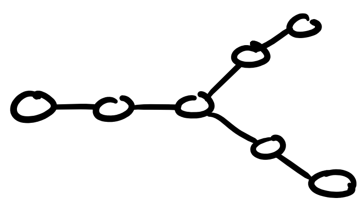
Can D be smaller than I_{max}?

Can D_{min} be smaller than I_{max}?

→ Yes 

So we have no guarantee of equality, and $|I| > |D|$ (possibly) by some arbitrary amount as demonstrated by star graphs

$$\Rightarrow \boxed{|D| \leq |I|}$$

(10) Define H: 

what we'll show:

we know from class

$H \nsubseteq T$ (not subgraph) ⇒ T is caterpillar ⇒ T is graceful

(weak induction proof prob. way easier)

Note: a tree is a caterpillar if all non-leaf vertices form a single path

→ this implies no $v \in V(T)$ can have more than 2 non-leaf have more than 2 non-leaf neighbors

→ If no such v exists, all non-leaf vertices have a max of 2 non-leaf neighbors

Consider subgraph induced on all non-leaf vertices in some T w/no H

→ must be connected, as we are effectively deleting all leaves, which can't disconnect a tree

→ max degree is 2, (from above) which gives us a path or ~~cycle~~ obviously not in a tree

⇒ so we have a single path, which has all leaves attached to it

⇒ T is caterpillar ⇒ T is graceful □