Graph Theory Midterm Practice Problems

Use these as study aids in conjunction with notes, homeworks, and weekly problems..

1. Consider the following in-degree and out-degree sequences for some hypothetical directed graph D. These sequences are not in any particular vertex order, so $S^+(1)$ and $S^-(1)$ don't necessarily refer to the same vertex. Construct a realization of G such that G has a directed Euler Tour.
   $S^+ = \{2, 4, 4, 6, 2, 3, 1\}, S^- = \{3, 2, 6, 4, 4, 2, 1\}$

2. Consider the following enumerative questions for undirected graphs, where loopy graphs and multi-graphs are proper supersets of simple graphs and we're considering a vertex set of cardinality n:

   (a) How many possible loopy graphs are there?

   (b) How many possible loopy multigraphs, with a maximum number of multi-edges of 2?

3. Construct a graph on 6 vertices that only has a single automorphism (called an *assymetric graph*).

4. Prove or disprove: Every tree with an even number of vertices has a perfect matching.

5. Consider the minimum-weight spanning tree T of some weighted simple connected graph G and some arbitrary $u, v \in V(G)$. Prove or disprove: a u, v-path in T has the minimum sum of edge weights of all possible u, v-paths in G.

6. Consider graph G, and prove that $\forall v \in G : G - v$ has a perfect match iff $|V(G)|$ is odd and $o(G - S) \leq |S| : \forall S \subseteq V(G)$.

7. Prove that every 3-regular graph has a perfect match if and only if it decomposes into P_4 subgraphs.

8. Prove that graph G with $\forall v \in V(G) : d(v) =$ even can be decomposed into cycle graphs.
   Hint: try induction on $|E(G)|$.

9. Consider minimum dominating set D and maximum independent set I on graph G. Provide and prove a tight bound for $|D|$ relative to $|I|$.

10. Prove tree T is graceful if it does not contain the below subgraph: