Graph Theory Weekly Problem 2

Due: 22 Jan 2026 at Midnight EST as a PDF on Submitty

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1. Consider a simple connected acyclic graph G with $|V(G)| \geq 1$. Using strong induction, show that $|E(G)| = |V(G)| - 1$. (4 pts)

   (a) First, determine an appropriate basis. Your basis must be in the same class $\mathbb{C}$ specified for the general graph $G \to \mathbb{C} = \{\text{simple, connected, non-null, acyclic}\}$. Generally, we often need our basis to be the smallest possible graph in $\mathbb{C}$. As we're trying to prove something related to edge cardinalities, we can take a decent guess that this will be our countable property for induction.

   (b) Note that we wish to proceed with a proof via strong induction. To do so, we will next consider some general $P(n) = G \in \mathbb{C}$, where $|E(P(n))|$ is greater than our basis. This is going to be our $P(n)$ case with $n = |E(P(n))|$.

   (c) To proceed, we need to figure out some *construction* on $P(n)$. This construction must be able to be applied to all possible $G \in \mathbb{C}$ (except for our basis), and it must result in some $P(k) \in \mathbb{C}$ where $|E(P(k))| = k < n = |E(P(n))|$. The most simple construction when considering edge cardinalities within strong induction is **edge deletion**, respresented as $P(n) - e$ for some $e \in E(P(n))$.

   (d) In order to invoke our induction hypothesis, we must show that some $P(k)$ resulting from our construction fits our initial graph class $\mathbb{C}$. The trick here, as is often the case with strong inductive proofs, is that $P(k)$ is not always equal to $P(n) - e$. In this instance, we can assume edge deletion on a tree always results in two subgraphs. Do these subgraphs fit our graph class $\mathbb{C}$? If so, we can use our induction hypothesis to assume that our proof statement holds. I.e., $|E(P(k))| = |V(P(k))| - 1$, with $P(k)$ representing one (or both) of these subgraphs.

   (e) To finish the proof, we 'undo' our construction and demonstrate how the proof statement we've assumed on $P(k)$ also holds on $P(n)$. Think about the relationship between our original $P(n)$, our construction, and the assumptions we've made on $P(k)$ with respect to the countable property on which we're performing the inductive proof.