Possible proofs (of many)

For all, we first consider w.l.o.g. a component of G where $|E| \geq |V|$ holds

→ One must obviously exist, as all cannot have $|E| < |V|$ while G has $|E(G)| \geq |V(G)|$

We can also assume G is simple

→ self loops and multi-edges are trivially cyclic

(1) Contrapositive

G has no cycles $\Rightarrow |V(G)| > |E(G)|$

Last WP we showed

$\{$connected acyclic simple$\} \Rightarrow |V| - 1 = |E|$
(i.e., is a tree)

Note: trees are <u>maximally acyclic</u>

→ adding an edge creates a cycle since all vertices are connected

$\Rightarrow$ so any acyclic graph must have at most as many edges as a tree

$\Rightarrow |E| \leq |V| - 1$ $\square$

$|E| < |V|$

② Strong induction  ← arm

Basis : $C_n \rightarrow |E| = |V| \rightarrow |E| \geq |V|$ ✓

(Note: if we use graph class $\mathcal{G}$, we need to show it holds $\forall F \in \mathcal{G}$)

Assume we have $P(n)$ with n edges and $n \geq |V(P(n))|$

Note: $\exists v \in V(P(n)) : d(v) = 1$

→ Otherwise, the minimum degree of G is 2 and it must have a cycle (proved in class via Extremal argument)

$P(k) = P(n) - v$

→ we delete exactly one vertex and edge, so $|E(P(k))| = |E(P(n))| - 1$

$|V(P(k))| = |V(P(n))| - 1$

and the inequality still holds

I.H. → $P(k)$ has a cycle

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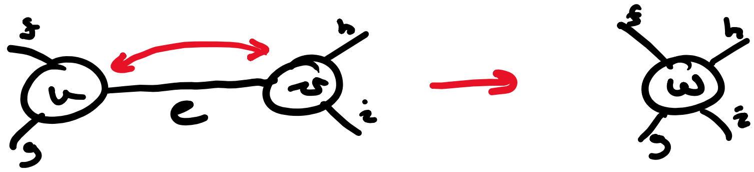
$P(k) + v \to P(n)$

→ Adding a vertex does not remove a cycle

$\Rightarrow P(n)$ is cyclic $\square$

③ Can do same proof above using <u>edge contractions</u> as construction



Basis could just be $P(1) = 8 = C_1$

④ Can do weak inductive proof on the contrapositive

Basis $P(0)$: ○ ← single vert $|V| > |E|$

I.H.: Assume $P(k)$ is acyclic and $|V| > |E|$

$P(k+1) = P(k) + e$

(Note our above assumption about the graph being connected)

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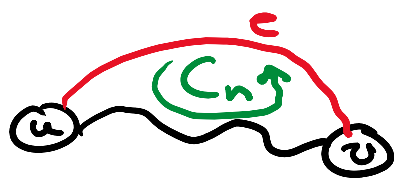
Consider the cases

1. e connects to 2 new vertices ✗
   → this makes $P(k+1)$ disconnected

2. e connects to 2 existing vertices ✗
   → this creates a cycle, as those 2 vertices are already connected



3. e must connect a new degree-1 (leaf) vertex to an existing vertex

$$\rightarrow |V(P(k+1))| = |V(P(k))| + 1$$
$$|E(P(k+1))| = |E(P(k))| + 1$$

$\Rightarrow$ and the inequality holds □