WP4 - Solution

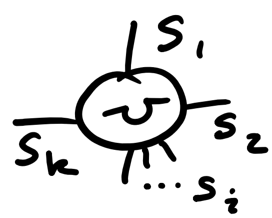
Saturday, February 7, 2026 7:18 PM

(1) Many different approaches:

- Direct structural argument
- Induction (can get wild and use k as our variable)
- Extremal argument
- Contrapositive?

Using direct structural argument:

- consider a vertex of max degree



called an S-lobe, $S = \{v\}$

- consider subgraph $\{S_i + v\}$ of the graph $\{T - v\}$
- It must have one leaf that isn't v (think of a max path from v, where must it end?)

EXXXXTREMAL!

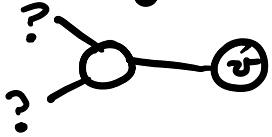
- Considering all k S-lobes, each

- Considering all k S-lobes, each must have at most 1 leaf

Q: Can we have a degree 3 vertex in an S-lobe?

→ Think of what this would imply using the logic above

OR Get Exxxtremal!



A: No, that would imply the S-lobe would need more than 1 leaf

=> Taken together, the structure of each S-lobe is simply a path from v to a leaf

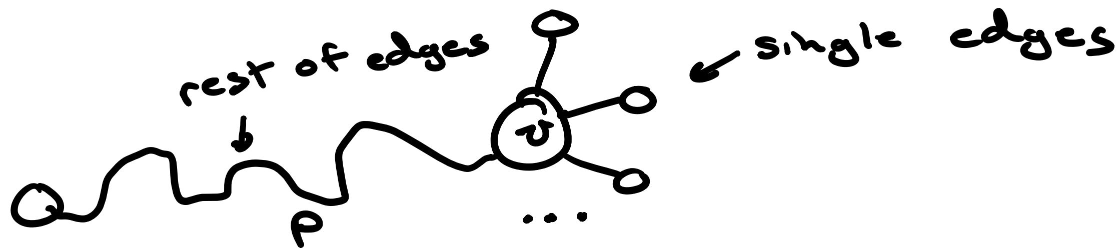
=> Hence T can be decomposed into k paths all sharing v $\square$

② Note that $|E| = |V| - 1$

Upper bound:

Upper bound:

we maximize diameter by maximizing the length of a single path from v



How many edges are in P?

$(k-1)$ in rest of graph

$|V|-1$ total

$|V|-1-(k-1) = |V|-k$

Diameter adds 1, since we traverse on final edge not in P

$$\Rightarrow \boxed{|V(T)|-k+1 \geq D(T)}$$

Lower Bound:

We minimize diameter by making all paths equal(ish) in length

$$|P_i| = \left\lceil \frac{|V|-1}{k} \right\rceil \text{ or } \left\lfloor \frac{|V|-1}{k} \right\rfloor$$

Diameter will be the length of the 2 largest paths

$$\boxed{D(T) \geq \left\lceil \frac{|V|-1}{k} \right\rceil + \left\{ \left\lceil \frac{|V|-1}{k} \right\rceil \text{ or } \left\lfloor \frac{|V|-1}{k} \right\rfloor \right\}}$$

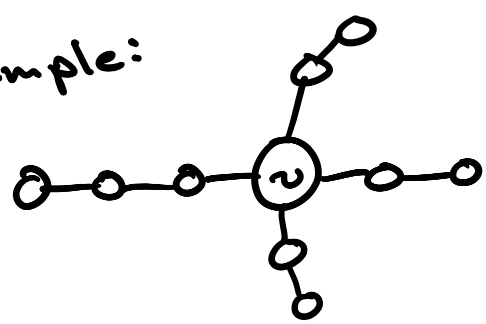
longest path

yes

next largest path, depends on whether $|V|-1 \bmod k > 1$

no

Example:



$k = 4$

$|V| = 10$

$\left\lceil \frac{10-1}{4} \right\rceil = 3$

$\left\lfloor \frac{10-1}{4} \right\rfloor = 2$

$D = 5$ ✓