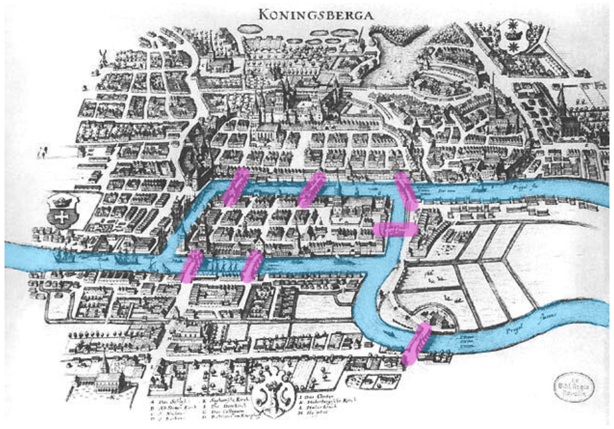
1.1 What are graphs?

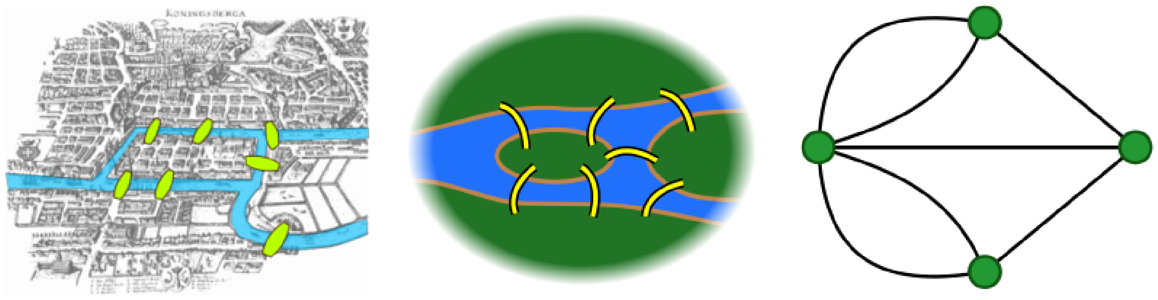
A graph is a mathematical structure of discrete entities (**vertices**) and the interactions between them (**edges**). Pretty much anything that exists can be described in the notation of graphs. Or, as I like to say, **everything is a graph**.

"Real-world" graphs are graphs that arise in biology, technology, social science, etc. Commonly studied graphs include road networks, social networks, protein interaction networks, the brain, the Internet, among many many others. Better understanding these real-world graphs has been a strong driver of the field of graph theory.

In fact, the field of graph theory arose hundreds of years ago when Euler contemplated the layout of seven bridges across a river within the city of Königsberg.



His question was whether it was possible to start on one side of the river, traverse each bridge exactly once, and then return to his original starting location. This problem can be easily modeled with a simple graph equivalent.



The question then becomes whether we can start at some vertex, traverse each edge exactly once, and then return to our starting vertex. Using some novel (but now pretty well-established) graph theoretic techniques, Euler was able to show that such a "tour" was in fact not possible on the given graph model. He was able to further prove the properties that must exist on a graph for an *Euler Tour* to exist. This was the first major result and the origination of the entire graph theory field. We'll dive into this proof and some of the major techniques used in upcoming classes.

1.2 Basic Definitions

Below are some basic (and more formalized) definitions and terminology that will be used throughout the course. As mentioned, graphs and aspects of graph theory appears within a wide variety of fields (computer science, math, network science, social sciences, physics, biology, many more), and all of these fields tend to use different notation and terminology. I'll try and keep things as consistent as possible for all of our sake. Additional terms and definitions will be introduced as we come across them.

A **graph** G is a tuple consisting of a set $V(G)$ of elements called **vertices**, a set $E(G)$ of pair of vertices called **edges**, and the **endpoint** relations of edges that associate each edge with two vertices. Additionally, each vertex and edge can have some non-zero number of *weights* associated with it (remember minimum spanning trees?). We consider **undirected graphs** for now, in which each edge is a non-directional pairwise relation.

If $e = (v, u)$ is an edge in G, then
e **joins** u and v; e is **incident** with u and v;
u and v are **incident** with e;
u and v are **endpoints** of e;
u and v are **adjacent** to each other;
u and v are in each others' **neighborhood**.
The **degree** of v is the number of unique $(v, u) \subseteq E(G)$ or simply the number of unique edges that have v as an endpoint.

The **order** of a graph $G(V, E)$ is $|V|$; the **size** of $G(V, E)$ is $|E|$. If both $|V|$ and $|E|$ are finite, G is called **finite**. A graph of order p and size q is called a (p, q)-graph.

Multiple edges or **multi-edges** are edges which have the same pair of endpoints. **Loops** are edges in which the endpoints are the same vertex.

A **simple graph** has no multiple edges or loops
A **null graph** is a graph with $V = E = \emptyset$
A **trivial graph** is a graph with $E = \emptyset$ and $|V| = 1$
An **empty graph** is a graph with $E = \emptyset$ and $|V| \geq 1$

A **path** is a simple graph whose vertices can be listed such that any two vertices are adjacent iff (if and only if) they are consecutive in the list. A **cycle** is a simple graph with an equal number of vertices and edges whose vertices can be placed around a circle and two vertices are adjacent iff they are appear consecutively along the circle. A **tree** is a simple connected graph with no cycles.

A **bipartite graph** is a graph which is the union of two disjoint independent sets.
A **complete graph**, or **clique**, is a graph where any two vertices in the graph are adjacent. We denote a clique of size n by K_n.
A **complete bipartite graph** or **biclique** with independent sets of sizes n and m we denote as $K_{n,m}$

A **subgraph** of a graph G is a graph H that is entirely **contained** in G ($H \subseteq G$), or that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ with all endpoint assignment being the same.

A bulk of this class will be diving deeper into proofs, algorithms, and interesting properties for each of the above various graph types (and more!). Some of the terminology used we haven't explicitly defined yet, but we will in due time as necessary.

1.3 Graph Representation

There are multiple ways to represent a graph. Below are a few examples.

An **adjacency matrix** $A(G)$ is an $n \times n$ (where $n = |V|$) matrix where a (positive) nonzero value in each $a_{i,j}$ indicates that many edges with endpoints from v_i to v_j. The sum of nonzeros in a row i is equal to the degree of v_i. For a simple graph with no loops, the diagonal will be zeros and the only nonzero appearing with be 1. For undirected graphs, the adjacency matrix will be symmetric.

An **incidence matrix** $M(G)$ is an $n \times m$ (where $n = |V|$ and $m = |E|$) in which a value of 1 in $m_{i,j}$ indicates that v_i is incident on edge e_j. Again the sum of nonzeros in a row i is the degree of v_i.

For memory efficiency with most real-world graphs, adjacency/incidence matrices are rarely used in computation. Note that most real-world graphs are extremely sparse. One of the easiest graph representations commonly utilized is to simply store for each vertex its degree and adjacencies (**adjacency format**). Another common representation is the **compressed sparse row** (CSR) format. It uses two arrays: the first array L of length $2|E|$ lists in order adjacencies of v_1 then v_2 then … v_n. The second array O of length $|V| + 1$ lists offsets for each v_i to where their adjacencies begin in the first array. The degree for any v_i can be calculated as $O[i+1] - O[i]$. For iterative computations, a CSR format can have a slight locality/cache benefit over the degree-adjacency representation.