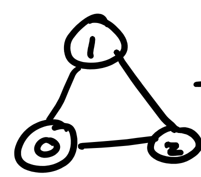
Lecture 3 - Walks, Connectivity, Induction

Friday, January 16, 2026 3:36 PM

During graduate school, Slota was only barred by his advisor from taking one single class.
That class: **Graph Theory**

Automorphism

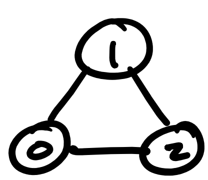
→ An isomorphic mapping of some G to itself
(s.t. the edge list is preserved)

↳ 

→ {(1,2), (1,0), (0,2)}

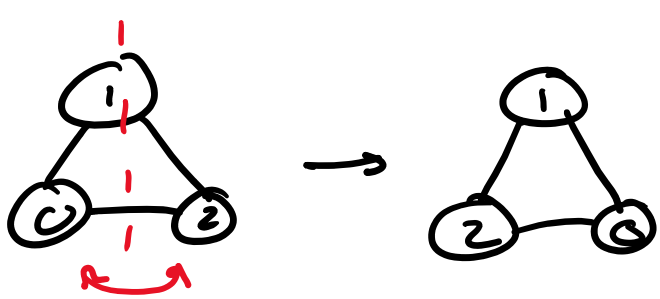
Consider some K_n:

→ we'll focus on K_3 for now

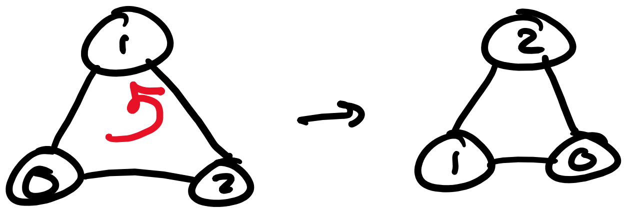


what are the automorphisms?

↳ think of automorphisms as the inherent symmetry in a graph

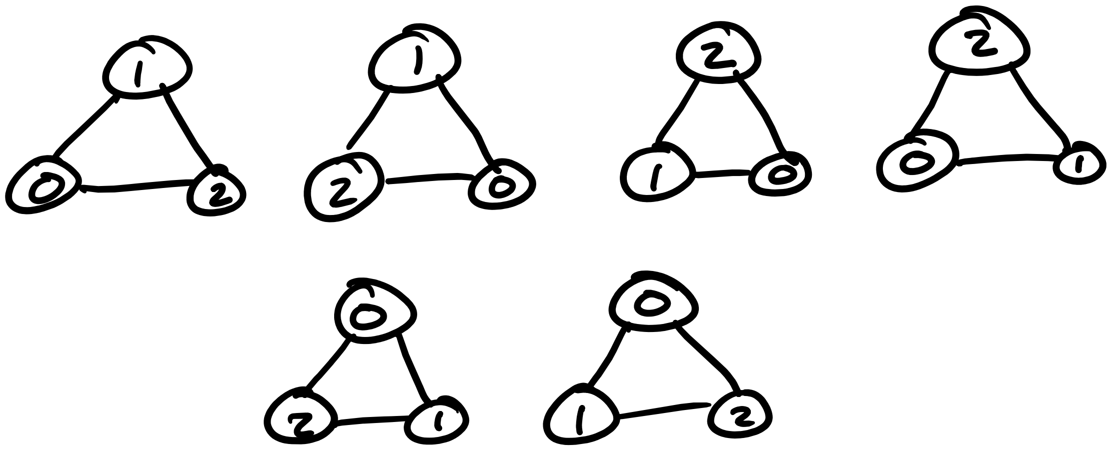


mirror symmetry



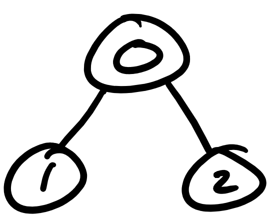
rotational symmetry

So how many automorphisms?



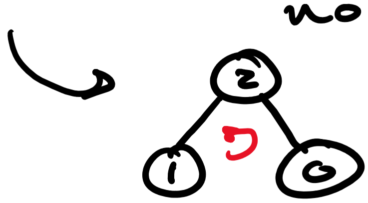
As every vertex can map to any other → $n!$ possible automorphisms

(this is the case for any clique K_n graph)

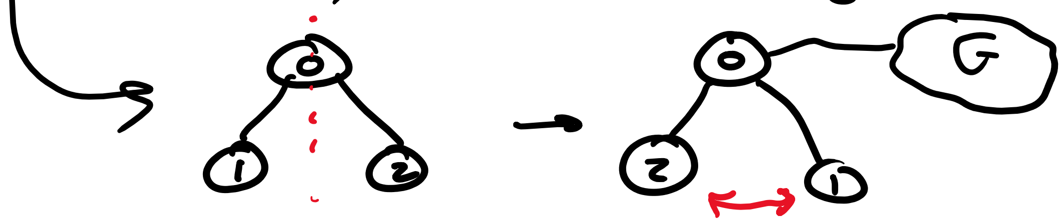


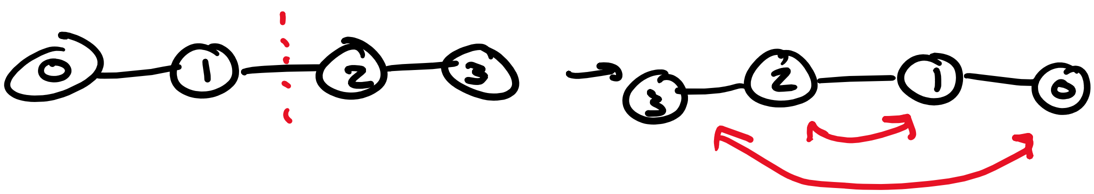
what about P_3?

— Rotational symmetry no longer applies



- And mirror symmetry?





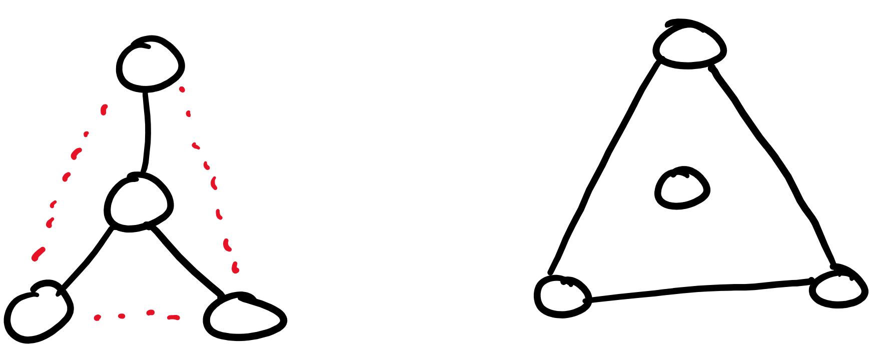
Even more definitions

<u>complement</u> of $G \rightarrow \overline{G}$

(assume G is simple)

$$V(\overline{G}) = V(G)$$

$$E(\overline{G}) = \{ \forall u,v \in V(G) : (u,v) \notin E(G) \}$$



$G \rightarrow \overline{G}$

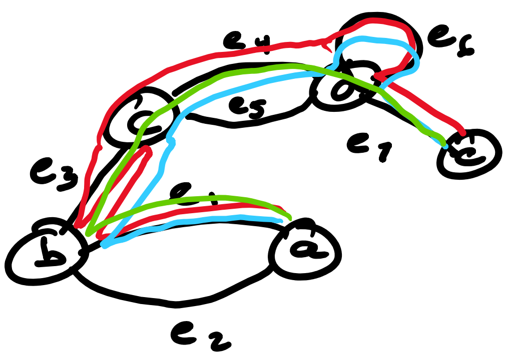
Time for a stroll

Time for a stroll

Walk: a list of edges (and vertices) s.t. each subsequent item in the list is adjacent/incident to the preceeding item

trail: as above, except edges do not repeat

path: as above, except edges and vertices do not repeat



W: $\{e_1\ e_3\ e_3\ e_3\ e_4\ e_6\ e_7\}$

(walk) (can call it a a,e-walk)

T: $\{e_1\ e_3\ e_4\ e_6\ e_7\}$

a,e-trail

P: $\{e_1\ e_3\ e_4\ e_7\}$

a,e-path

A walk/trail/path that starts and end at the same vertex is called closed

Note: a closed path is really

Note: a closed path is really a cycle

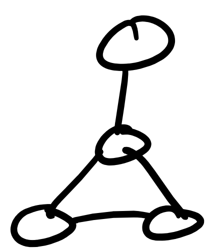
Length: number of edges traversed

Hop: traversal of a single edge

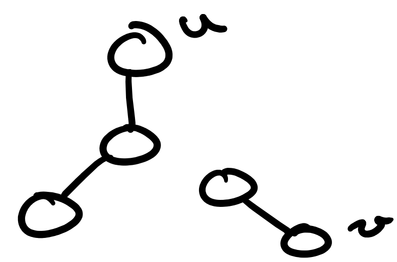
Let's get connected

Recall: G is connected if

$$\forall u,v \in V(G): \exists\, u,v\text{-path}$$



G is connected

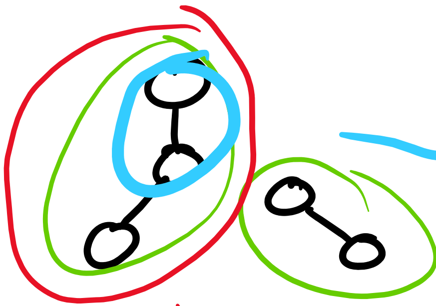


G' is disconnected

connected component: a **maximal** connected subgraph of some G

maximal: can not be made larger

maximum: the largest possible



2 connected components

connected subgraph (not maximal)

maximum connected subgraph

Note: same for **minimal** / **minimum**

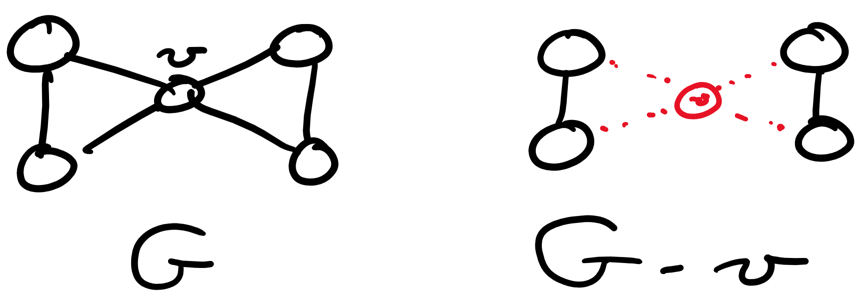
but smaller / smallest

cut vertex: some $v \in V(G)$

s.t. $\underline{G - v}$ has more components than G

vertex deletion

Remove v and all edges incident on v



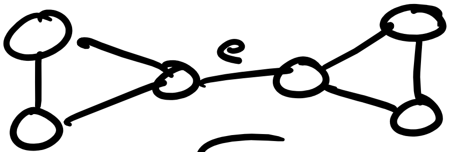
cut edge: some $e \in E(G)$

s.t. $\underline{G - e}$ has more comps.

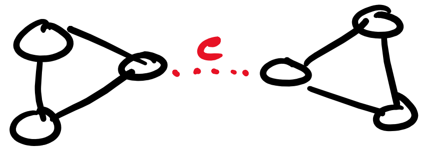
s.t. $G-e$ has more comps.

↳ edge deletion

only remove edge e



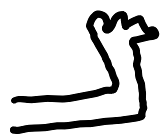
G



$G-e$

Time to induce

(aka Big Dog of G.T. #1)

weak induction 

Prove: $2^1+2^2+\ldots+2^n = 2^{n+1}-2$

Basis: $P(1) \rightarrow 2^1 = 2^{1+1}-2 = 4-2 = 2$ ✓

Inductive Step: $P(n=k+1)$

Inductive Hypothesis: we assume what we're trying to prove holds for some $P(k)$

→ show it holds for $P(k+1)$

$$P(n=k+1) = \underbrace{2^1+2^2+\ldots 2^k}_{\text{I.H.} \Rightarrow 2^{k+1}-2} + 2^{k+1}$$

$$= 2^{k+1} - 2 + 2^{k+1}$$

$$= 2^{k+2} - 2$$

$$= 2^{(k+1)+1} - 2$$

$$= 2^{n+1} - 2 \quad \square$$

weak induction

$P(1)$, $P(2)$ ... $P(k)$, $P(k+1)$ ...

basis (could be $P(0)$, $P(2)$, etc.)

assume holds via I.H.

show it holds given I.H.

↳ we prove it holds for $P(1) \ldots P(\infty)$

strong induction

$P(1)$, ..., $P(k)$, ..., $P(n)$ ....

basis

assume holds via I.H. for $1 \leq k < n$

show it holds

via I.H. for

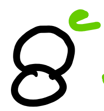
$1 \le k < n$

Example Proof:

length is odd

Show every closed odd walk contains an odd cycle

We'll use strong induction on the length of the walk

Basis $P(1)$:  ✓

Inductive step:

Assume we have some $P(n)$, where $P(n)$ is an odd closed walks

Consider the cases

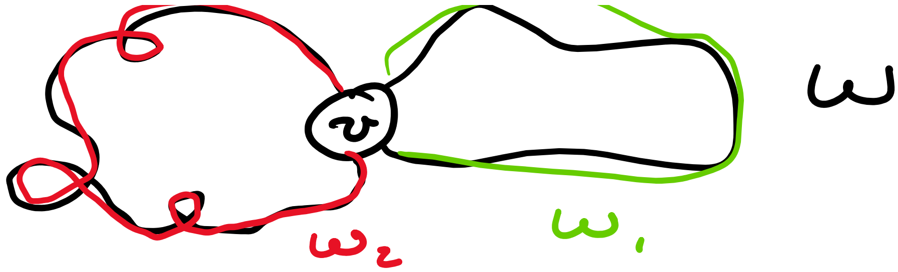
Generally: what possible configurations we can use to simplify our proof

Case 1: no vertices repeat

$\Rightarrow$ trivially, $P(n)$ is a cycle

Case 2: at least one vertex repeats

w



Assume v is a vertex that repeats

→ Note that we can separate ω into ω_1 and ω_2

% % %

integer parity

% %

↳ this means odd + odd = even

even + even = even

odd + even = odd

As $|\omega| = |\omega_1| + |\omega_2|$

odd = odd + even

this implies w.l.o.g. that $|\omega_1|$ = odd

(without loss of generality)

Note: $|\omega_1| < |\omega|$

$|\omega_1|$ is odd

ω_1 is closed

☆ ω_1 fits our original class under consideration

☆ W_1 fits our original class under consideration

Our construction

From $P(n) \to P(k)$
where $1 \leq k < n$

- we take $P(n)$ and split it around v, which is a repeated vertex
- then we take $P(k)$ as a subwalk that is odd

$\Rightarrow P(k) = W_1$ as above

We can use our I.H. on $P(k)$

$\to P(k)$ contains an odd cycle

To finish, undo our construction and show our property assumed on $P(k)$ holds on $P(n)$

In this case:

Adding W_2 back with W_1 does not affect the odd cycle

not affect the odd cycle
that exists on W_1

$\Rightarrow$ so an odd cycle exists
on $P(n)$ $\square$