Lecture 4 - Proof Techniques and Eulerian Circuits

Wednesday, January 28, 2026 4:47 PM

Snow tires vs. all seasons
- Softer material -> better grip in cold weather
- Deeper tread -> better grip in soft snow
- Sipes (little squiggles) -> better grip on ice and hard snow
- Stopping distance up to 50% shorter, much better handling

All weather tires vs. all seasons
- Have deeper tread and usually sipes, firmer rubber to last longer

Cost considerations
- Cost of two sets of tires mostly mitigated to longer life for each
- Used rims, junkyards (note: most vehicles can fit multiple rim sizes)
- Need to get tires mounted, but can swap out rims on your own (RSAS)

Last class:

G has closed odd walk

$\Rightarrow G$ has odd cycle

$\mathcal{G}$ = class of graphs (subgraphs)

$\mathcal{G}_1 = \{\text{closed odd walk}\}$

$\mathcal{G}_2 = \{\text{odd cycle}\}$

What we're really showing

$$\mathcal{G}_1 \Rightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_1 \underset{\text{subset}}{\subseteq} \mathcal{G}_2$$

any $G \in \mathcal{G}_1 \Rightarrow G \in \mathcal{G}_2$

To approach any graph proof:

To approach any graph proof:

Consider the properties that define each graph class under consideration

For our current proof:

walk: vertices, edges can repeat

closed: start/end at same vertex

odd: think: parity

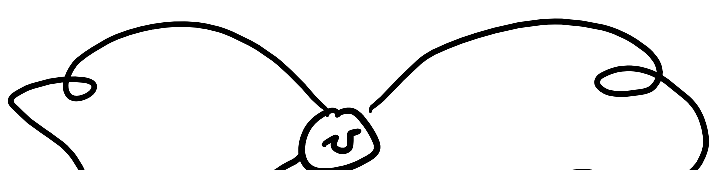
We can often simplify arguments by considering subclasses of our G_1

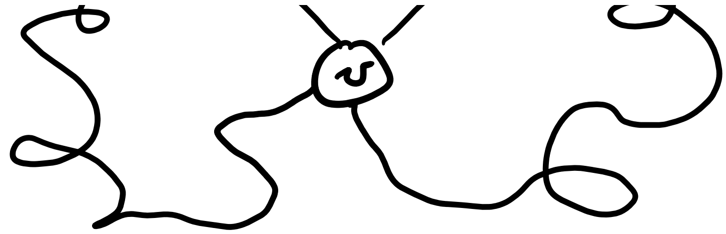
E.g., 1. No repeated vertices

2. Yes repeated vertices

→ must encompasses all G_1 (sp?)

We also usually want to explicitly consider structure/topology





Useful: draw it out

→ here we can split our w into some w_1 and w_2

Note: inductive proofs are inherently recursive

→ our arguments and construction must recurse back to our basis or another sub-case

Note 2: Our construction must capture *all* possible topologies in the class/subclass being considered

→ this is where weak induction often breaks down

And: our w_1 produced from our construction must be $w_1 \in G_1$ in order for us to use I.H. (inductive hypothesis)

Proof Techniques

* Basic structural arguments
  - → basis for most logical statements

* Consider the cases
  - → simplify trivial cases
  - → simplify topologies to consider

* Parity arguments
  - → many graph properties are countable

* Pigeonhole principle
  - → We have x items and y buckets where $x > y$
  - → At least 2 items in 1 bucket

* Extremal principle
  - → Given a set of countable and orderable items (finite/well orderable)

(finite | well orderable)

→ $\exists$ some item with a maximum value in that set
and $\exists$ some item with a minimum value in that set

$S = \{1, 0, \underline{10}, 9, -1, \underline{-5}\}$

max min

* Necessity and sufficiency
aka: equivalence relations

$C_1 = \{\text{property } A\}$

$C_2 = \{\text{property } B\}$

show $C_1 \Leftrightarrow C_2$
equivalence

To prove:

show property A ⇒ property B
then show property B ⇒ property A

* Contrapositive

if property A => property B

→ not property B => not property A

G is bipartite => G has no odd cycles

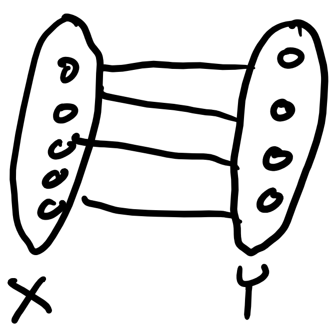
G has odd cycles => G is not bipartite

Prove: G has no odd cycles

$\iff$ G is bipartite

First show ($\Leftarrow$)

G is bipartite => G has no odd cycles



Note: cycle can be considered a closed path

Consider all possible paths from some $x \in X$

We can only traverse from some $u \in X$ to $v \in Y$ and from some $v \in Y$ to $u \in X$

⇒ any odd path from x will end up at some $y \in Y$

=> no odd cycle can exist ✓

Now, the other direction ($\Rightarrow$)

G has no odd cycles $\Rightarrow$ G is bipartite

w.l.o.g. assume G is connected

(note that any arguments can simply be applied to any component of G if not)

Consider some $v \in V(G)$

define: $f(u)$ where $u \in V(G)$

$f(u)$ = shortest path from v

define: $X = \{\forall x \in V(G) : f(x) = \text{even}\}$

$Y = \{\forall y \in V(G) : f(y) = \text{odd}\}$

Q: are X and Y <u>independent</u>?

no edge between vertices in set

Note: $X \cap Y = \emptyset$

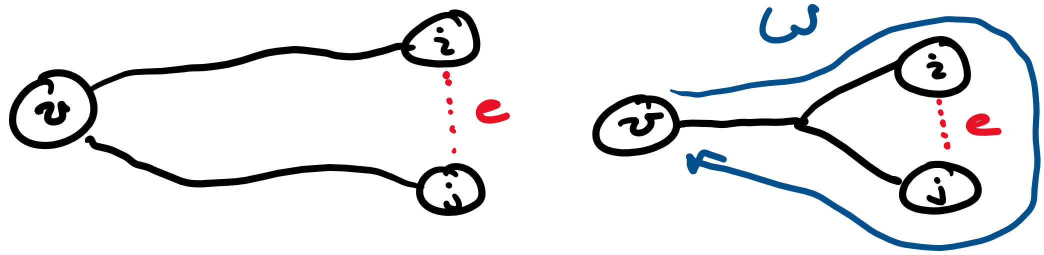
intersection   empty set

Consider two vertices in X or Y

$i, j \in X$ or $i, j \in Y$

Consider shortest v,i-path & v,j-path

Consider shortest v,i-path & v,j-path



Q v2: can edge (i,j) exist?

Consider a walk W

$W = \{v \text{ to } i,\ (i,j),\ j \text{ to } v\}$

following shortest paths

We want show if W can exist

Note: parity

Consider the length of W:

odd + odd + 1 = odd

even + even + 1 = odd

From our prior proof:

A closed odd walk contains an odd cycle

an odd cycle

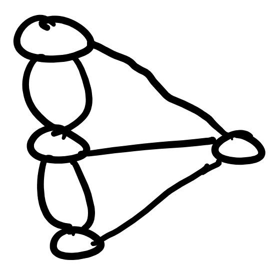
However: we assume we have no odd cycles

=> this implies that (i,j) can not exist

=> Our definition of X, Y defines a pair of bipartite sets

=> our graph is bipartite $\square$

Recall Euler and his bridges

as a graph 

Euler: does a closed trail exist that traverses all edges?

→ Euler tour / circuit / trail

We wish to "characterize" an Euler Tour

→ define the properties of graphs

→ define the properties of graphs that contain an Euler Tour

First: show if $\forall v \in V(G) : d(v) \geq 2$

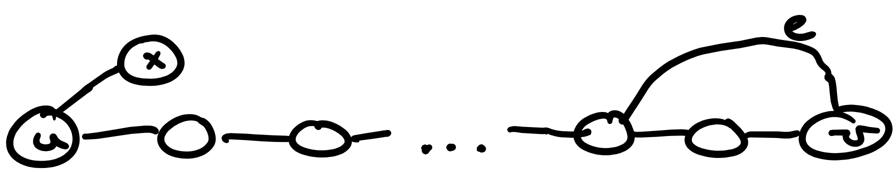
$$\Rightarrow \exists C_n \subseteq G$$

To due this: **Extremal principle**

Consider $P \subseteq G$
path on G

s.t. P is the maximum length

consider P's structure



w.l.o.g.: can vertex x exist?
$x \in N(u) : x \notin P$

Answer: No, if $x \notin P$ we can extend P by adding x

Contradiction

Contradiction

on our selection of P

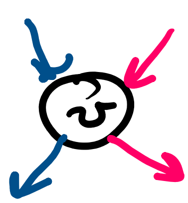
↳ so x as defined it cannot exist

$\Rightarrow$ we have an edge from u to some $w \in P$

$\Rightarrow$ this creates a cycle $\square$

What are the <u>necessary</u> properties of a G with an Euler Tour??

1. G has at most 1 nontrivial component
2. $\forall v \in V(G): d(v) = \text{even}$



every time we reach a vertex on our trail, we must exit via some other edge

Q: are these necessary conditions are also sufficient?

are also sufficient?

G is Eulerian $\Leftrightarrow$ $\forall v \in V(G): d(v) = \text{even}$ and G has at most 1 nontrivial component