Lecture 5 - Degrees, Graphic Sequences, Digraphs

Monday, February 2, 2026 8:35 AM

SLPT (Slota life prop tip): If you forget your RPI ID card, don't try and sneak into the Mueller Center. They will call PubSafe and you will have to hide in a bathroom stall to escape a swarm of safety officers.

Last class: we show necessity

→ now: sufficiency

G has at most 1 nontrivial component

& $\forall v \in V(G): d(v) = \text{even}$

$\Rightarrow$ G is Eulerian

We'll do strong induction on $|E|$

Basis: $P(0) \rightarrow$ trivial tour $\{\}$

Assume we have $P(n) \in \mathcal{G}$ ← class specified above

(we'll only consider nontrivial comp.)

Note: minimum degree is at least 2

→ from last class: $\exists C_n \in G$

$P(k) = P(n) - C_n$

Note: deleting a cycle subtracts 2 degrees from each $v \in V(C_n)$

Note 2: P(k) might be disconnected

Note 2: $P(k)$ might be disconnected
→ consider each $P(k)$ component

Note 3: $P(k)$'s nontrivial components
are all in G

I.H. → each $P(k)$ component has
an Euler Tour

Q: How can we use this to prove
Eulerianness on $P(n)$?
(word?)

PROOF BY ALGORITHM

↳ we construct an algorithm that proves
or demonstrates some property

To complete our proof: algorithmically
combine $P(k)$'s sub-tours with
C_n to create a tour on $P(n)$

Our Algorithm:

Start at some $v \in V(C_n)$

Start at some

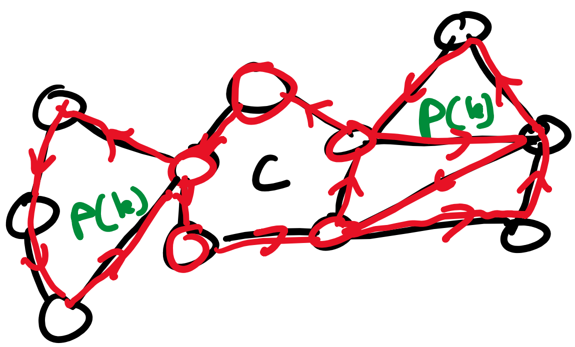
if $d(v) = 2$:

traverse on C_n

else:

$\exists$ a tour starting/ending at v through a component of $P(k)$

continue along C_n repeating the above, traversing edges in tours for components of $P(k)$ that have not already been traversed



$\square$

Degrees

recall: $n = |V(G)|$

$m = |E(G)|$

degree of $v \rightarrow d(v)$ or d_v

for simple graphs $d(v) = |N(v)|$

For graph G:

maximum degree: $\Delta(G)$

minimum degree: $\delta(G)$

G is k-regular if:

$$\delta(G) = k = \Delta(G)$$

$$\forall v \in V(G): d(v) = k$$

Examples: C_n is 2-regular

K_n is $(n-1)$-regular

Degree sum formula:

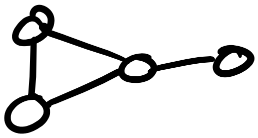
$$\sum_{v \in V(G)} d(v) = 2m = 2|E(G)|$$

why: each edge adds +1 to degrees to the degrees of each of the 2 endpoints

Big Q: what are possible degrees for some graph G?

Degree sequence: list of all degrees

Degree sequence: list of all degrees for vertices in some G

G  $S = \{3, 2, 2, 1\}$

NEW Underlying assumption:

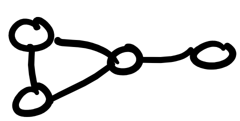
G is <u>simple</u>

<u>Graphic sequence</u>: a list of degrees that can <u>realize</u> a simple undirected graph

<u>Realize</u>: construct a graph with a given degree sequence

$S = \{1, 3, 2, 2\}$

↳ realize?

↳ 

↳ S is graphic

Q: What sequences are graphic?

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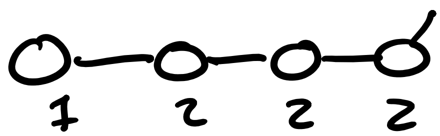
$S_1 = \{1, 2, 2, 2\}$

↳ degree sum formula

odd sum → can't realize

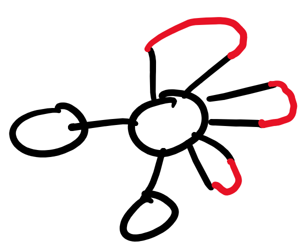


← stub, can't happen

Takeaway → an even degree sum is a <u>necessary</u> condition

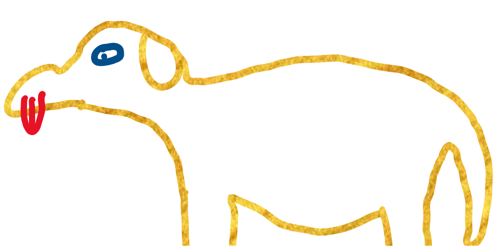
Q: is it <u>sufficient</u>?

$S_2 = \{8, 1, 1\}$ → even ✓

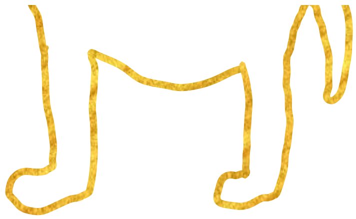


← lotta stubs

So how can we actually determine if some S is graphic?



Havel-Hakimi



Havel - Hakimi Theorem (/algorithm)

Big dog of GT

Given a non-increasing sequence

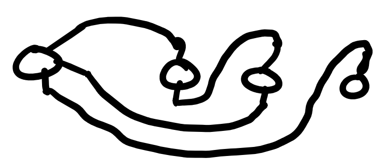
$$S = \{d_1, d_2, d_3, \ldots, d_n\}$$

$$d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$$

S is graphic iff (if and only if)

$$S' = \{(d_2 - 1), (d_3 - 1), \ldots, (d_{(d_1+1)} - 1), \ldots, d_n\}$$

is graphic



Example: $S = \{\cancel{3}, \underset{-1}{2}, \underset{-1}{2}, \underset{-1}{1}\}$

$S' = \{\cancel{1}, \underset{-1}{1}, 0\}$

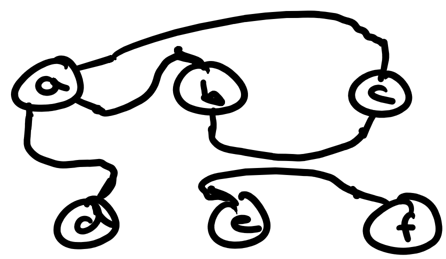
$S'' = \{0, 0\}$ ← trivially graphic

Ways we can tell if some S^i is not graphic via this process:

is not graphic via this process:

1. we end up with a single nonzero value
2. we end up with negative value(s)
3. we have some d_1 that is larger than the number of other nonzeros in the sequence

$$S = \{\overset{a}{\cancel{3}}, \underset{-1}{\overset{b}{2}}, \underset{-1}{\overset{c}{2}}, \underset{-1}{\overset{d}{1}}, \overset{e}{1}, \overset{f}{1}\}$$



$$S' = \{\overset{b}{1}, \overset{c}{1}, \overset{d}{0}, \overset{e}{1}, \overset{f}{1}\}$$

$$S' = \{\overset{b}{\cancel{1}}, \underset{-1}{\overset{c}{1}}, \overset{e}{1}, \overset{f}{1}, \overset{d}{0}\}$$

$$S'' = \{\overset{e}{\cancel{1}}, \underset{-1}{\overset{f}{1}}, 0, 0\}$$

$$S''' = \{0, 0, 0\}$$

↳ trivially realizable

Havel-Hakimi Algorithm

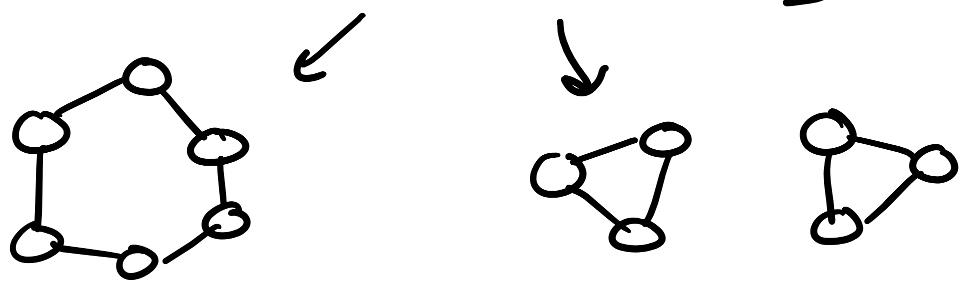
1. Map each value in S to some vertex
2. we draw an edge (d_i, d_j) when some d_j is decremented by the removal of d_i

the removal of d_i

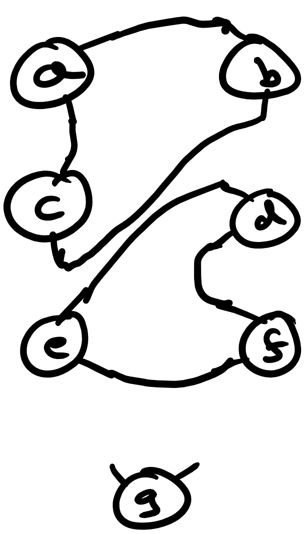
3. Iterate until S^* is all zeros

Note 1: a graphic sequence does not necessarily have a unique realization

$$S = \{2, 2, 2, 2, 2, 2\}$$



Note 2: It is necessary to sort



$$S = \{\overset{a}{\cancel{2}}, \underset{-1}{\overset{b}{2}}, \underset{-1}{\overset{c}{2}}, \overset{d}{2}, \overset{e}{2}, \overset{f}{2}, \overset{g}{2}\}$$

$$S' = \{\overset{b}{\cancel{1}}, \underset{-1}{\overset{c}{1}}, \overset{d}{2}, \overset{e}{2}, \overset{f}{2}, \overset{g}{2}\}$$

$$S'' = \{\overset{c}{\cancel{0}}, \overset{d}{\cancel{2}}, \underset{-1}{\overset{e}{2}}, \underset{-1}{\overset{f}{2}}, \overset{g}{2}\}$$

$$S''' = \{\overset{e}{\cancel{1}}, \underset{-1}{\overset{f}{1}}, \overset{g}{2}\}$$

$$S'''' = \{0, 2\}$$ × not graphic

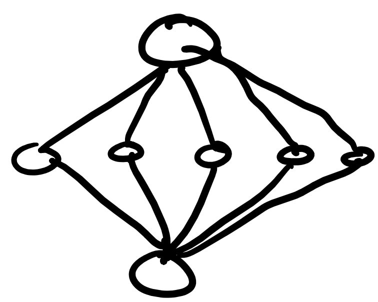
Braincercize: Can all possible realizations for a given graphic sequence be constructed via Havel-Hakimi?

Proof by counter-example

$S = \{5, 5, 2, 2, 2, 2, 2\}$

Note: via Havel-Hakimi, vertices with degree of 5 will always be connected

However:



=> **NO**

One final thing:

We can generate multiple different realizations by swapping degree-vertex pairs during our sorting

i.e., our sorting does not need to be stable

..., but sorting does not need to be <u>stable</u>

$S = \{ \overset{a}{\cancel{2}}, \overset{b}{2}, \overset{c}{2}, \overset{d}{2}, \overset{e}{2}, \overset{f}{2} \}$

$\qquad\quad -1 \quad -1$

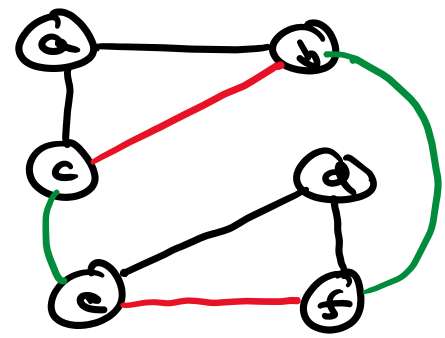
$S' = \{ \overset{d}{\cancel{2}}, \overset{e}{2}, \overset{f}{2}, \overset{b}{1}, \overset{c}{1} \}$

$\qquad\quad -1 \quad -1$

$S'' = \{ \overset{e}{1}, \overset{f}{1}, \overset{b}{1}, \overset{c}{1} \}$

$S'' = \{ \overset{b}{1}, \overset{f}{1}, \overset{c}{1}, \overset{e}{1} \}$

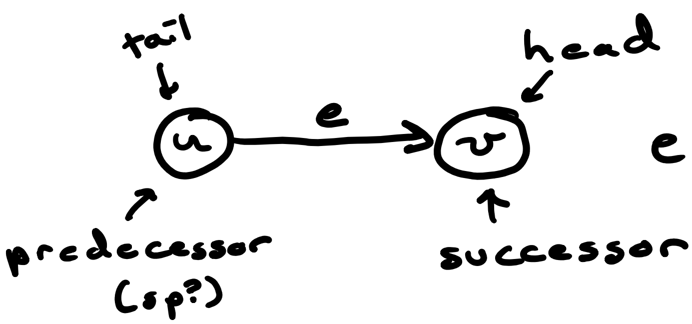
Proof by example



Directed graphs

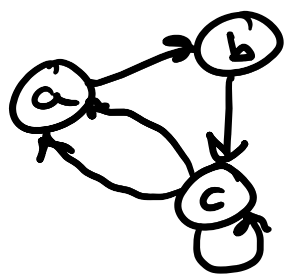
aka digraphs

→ graphs with directed edges



$e = (u, v)$

Adjacency matrix is not necessarily symmetric anymore symmetric anymore



D

$$A = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & 0 & 0 & 1 \\ c & 2 & 0 & 1 \end{array}$$

sum is out degree

sum is in degree

For digraphs, we consider <u>out-degrees</u> and <u>in-degrees</u> separately

<u>out-degree</u> $= d^+(v) =$ number of edges pointing from v

<u>in-degree</u> $= d^-(v) =$ number of edges pointing to v

We equivalently have:

$\delta^-(G) =$ min in degree on G

$\delta^+(G) =$ min out degree

$\Delta^-(G) =$ max in degree

$\Delta^+(G) =$ max out degree

And one final thing:

directed degree sum formula

directed degree sum formula

$$\sum_{v \in V(D)} d^+(v) = |E(D)| = \sum_{v \in V(D)} d^-(v)$$