

7.1 Tree Enumeration

Cayley's Formula states that with a vertex set of size n there are n^{n-2} possible trees. What this means is that there is n^{n-2} ways to form a list of length $n - 2$ with entries created from a given vertex set. A list for a specific tree is its **Prüfer code**. For a given tree T with some logical ordering of vertex identifiers, we can create its **Prüfer code** by first deleting the lowest value leaf and outputting that leaf's neighbor as a value in our code. We can also use a given vertex set S and a **Prüfer code** to recreate T . See the proof in the book that for a set $S \in \mathbb{N}$ of size n , there are n^{n-2} trees with vertex set S .

Below are the algorithms for creating a **Prüfer code** from T and recreating T from a **Prüfer code**.

```

procedure CREATEPRUFER(Tree  $T$  with vertex set  $S$ )
     $a \leftarrow \emptyset$                                 ▷ Initialize Prüfer code to null
    for  $i = 1 \dots (n - 2)$  do
         $l \leftarrow$  least remaining leaf in  $T$ 
         $T \leftarrow (T - l)$ 
         $a_i \leftarrow$  remaining neighbor of  $l$  in  $T$ 
    return  $a$ 

```

```

procedure RECREATETREE(Prüfer code  $a$  created with vertex set  $S$ )
     $V(T) \leftarrow S$                                 ▷ Tree has vertex set  $S$ 
     $E(T) \leftarrow \emptyset$                             ▷ Initialize tree edges as empty
    initialize all vertices in  $S$  as unmarked
    for  $i = 1 \dots (n - 2)$  do
         $x \leftarrow$  least unmarked vertex in  $S$  not in  $a_{i \dots (n-2)}$ 
        mark  $x$  in  $S$ 
         $E(T) \leftarrow (x, a_i)$ 
     $x, y \leftarrow$  remaining unmarked vertices in  $S$ 
     $E(T) \leftarrow (x, y)$ 
    return  $T$ 

```

How many different ways can we create a graph given a vertex set of size n ? **Cayley's Formula** states that with a vertex set of n there are n^{n-2} possible trees. A **spanning subgraph** of some graph G is a subgraph that contains all vertices in G . A **spanning tree** is a spanning subgraph that is also a tree. Another way to think about Cayley's Formula: the number of possible trees is the number of possible spanning tree configurations of complete graph. How might we compute the number of spanning trees of a general graph?

We can use a simple recurrence relation to do so. The number of possible spanning trees in a graph $\tau(G)$ is equal to the sum of the number of spanning trees of the graph with

an edge removed $\tau(G - e)$ plus the the number of spanning trees of the graph with an edge contracted $\tau(G \cdot e)$. An **edge contraction** involves combining the endpoints u, v of a given edge e into a single vertex, such that the new vertex has incident edges of all original edges of both u and v except for e . We'll see more on recurrence relations in the future.

7.2 Graceful Labeling

A **graceful labeling** of a graph is a labeling of all n vertices of a graph with unique labels from $0 \dots m$, such that each of the m edges has a unique value computed as the difference between the labels of its endpoints. A graph is **graceful** if it has a graceful labeling.

Ringel-Kotzig Conjecture: all trees are graceful. This is unproven, however, certain subsets of of trees have been proven to be. These include paths and **caterpillar graphs**. Caterpillar graphs are trees in which a single path is incident to or contains every edge in the graph. Proving this conjecture will guarantee you an A in this course (though I get to be a co-author on the paper).