Celery, an unsung hero?

- French Mirepoix: carrots, onion, **celery**
- Cajun Holy Trinity: onions, peppers, **celery**
- Italian Soffritto: onions, carrots, **celery**
- Many many soups: chicken noodle, minestrone, khoresh karafs
- Victorian era: considered a luxury food, special display vases for it
  - Heirloom varieties less hardy for growing and transport
  - Probably tasted a lot different
- ~100 years ago in NYC:
  - Third most popular restaurant item after coffee+tea
  - Served raw, chilled, and salted

Poll: Move HW3 deadline to Sunday night with no late days, but drop guarantee that they will be graded before the exam?

Yes: Most No: 1

General Graph Matching

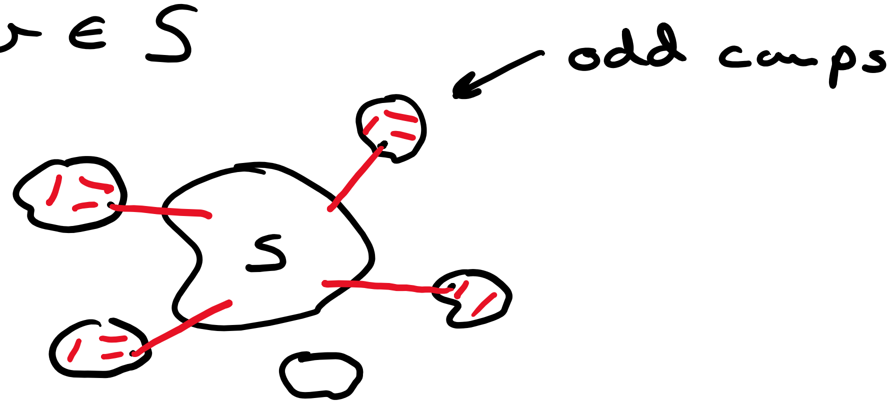
$o(G)$ = # of odd components of G
↳ # of odd vertices

Tutte: G has a perfect match iff
$$\forall S \subseteq V(G): o(G-S) \leq |S|$$

($\Rightarrow$) consider some S and $G-S$

Note: odd comps of $G-S$ cannot have a perfect match

↳ at least one vertex is each must be matched to some $v \in S$



$\Rightarrow |S|$ must be bound below by $o(G-S)$

$\forall S \subseteq V(G): o(G-S) \le |S| \Rightarrow G$ has P.M.

contrapositive

G has no P.M. $\Rightarrow \exists S \subseteq V(G): |S| < o(G-S)$

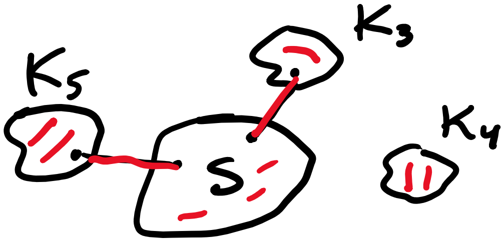
Note: condition holds if we add edges to some edges to same

Extremal

We will consider an extremal choice of $G \to G'$, where G' is edge-maximal w.r.t. no P.M.

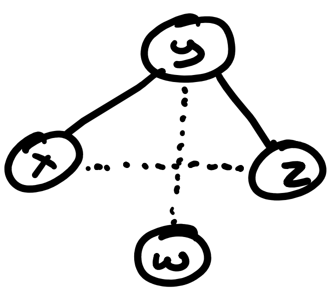
Define $S = \{ \forall v \in V(G) : d(v) = |V(G)| - 1 \}$

Case 1: $G' - S \to$ all components are cliques



Note: S must be "bad" $|S| < o(G-S)$ as we can otherwise trivially construct P.M.

Case 2: $G' - S \to$ not all cliques



$\exists\, x, z$ s.t. $(x,z) \notin E(G'-S)$

$\exists\, y$ s.t. $(x,y)(z,y) \in E(G'-S)$

$\exists\, w$ s.t. $(y,w) \notin E(G'-S)$

From our selection of G'

s $G' + (u,w)$ creates a P.M.

$$\begin{cases} G' + (y,w) \text{ creates a P.M.} \\ G' + (x,z) \text{ creates a P.M.} \end{cases}$$

→ we'll show this actually implies a P.M. on G itself

Define:

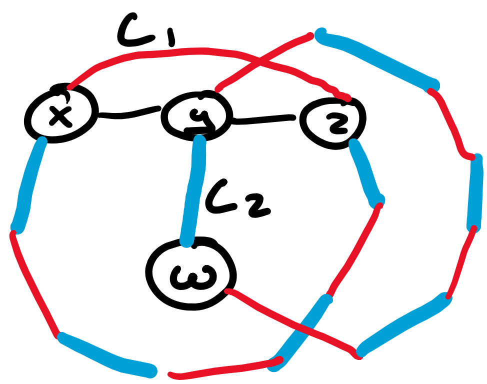
M_1 = P.M. on $G' + (x,y)$

M_2 = P.M. on $G' + (y,w)$

$F = M_1 \Delta M_2$ → must be paths or cycles

However: as M_1 and M_2 are P.M.s, we only have cycles

M_1 M_2



C_1 = cycle w/ (x,z)

C_2 = cycle w/ (y,w)

Case 2a: $C_1 \neq C_2$

P.M. on G' = all $e \in M_2$, $e \in C_1$

all other M_1

all other M_1

$\hookrightarrow$ P.M. without using (y,w) or (x,z)

contradiction

$\rightarrow$ S must be "bad" ✓

Case 2b: $C_1 = C_2$

P.M. on $G' = M_1$ on C_2 from w until x or z

if we reach x:

P.M. on $G' \mathrel{+}= (x,y) + M_2$ from y to z

if we reach z:

P.M. on $G' \mathrel{+}= (y,z) + M_2$ from y to x

$\rightarrow$ either way we have a P.M. w/o (x,z) or (y,w)

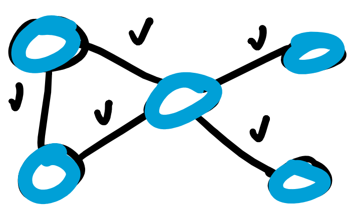
contradiction

$\rightarrow$ S must be bad ✓

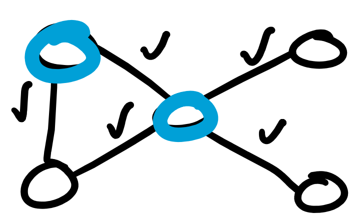
$$\boxed{\begin{array}{c}\text{Tutte: } G \text{ has a P.M.} \\ \text{iff} \\ \forall S \subseteq V(G):\ o(G-S) \leq |S|\end{array}}$$

Vertex and edge covers

<u>vertex cover</u>: a set $Q \subseteq V(G)$ that has at least one endpoint on $\forall e \in E(G)$



$\rightarrow$

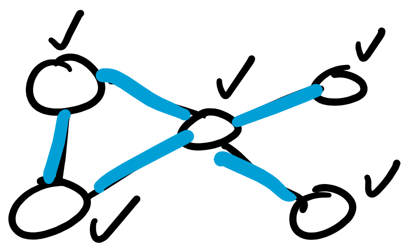


V(G) is trivially a cover

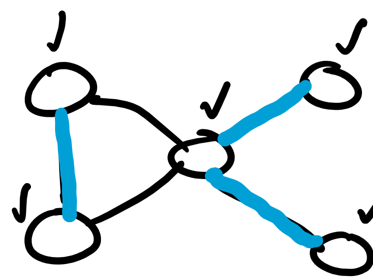
minimum cover

<u>edge cover</u>: a set $C \subseteq E(G)$ that has at least one

that has at least one edge incident on $\forall v \in V(G)$



$E(G)$ is trivially an edge cover



minimum edge cover

Note: lower bound on $|C|$ is $|C| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil$

König-Egerváry: on a bipartite graph, the size of a minimum vertex cover is equal to the size of a max match

$|M_{max}|$ = max match

$|Q_{min}|$ = min vert cover

K.E.: $|M_{max}| = |Q_{min}|$ on bigraph G_{XY}

K.E.: $|M_{max}| = |Q_{min}|$ on bigraph G_{XY}

Note: $\forall v \in \underset{\text{cover}}{Q}$ at most one $e \in \underset{\text{match}}{M}$

can be covered

$\to |Q| \geq |M|$ in general

$\Rightarrow |Q_{min}| \geq |M_{max}|$

and $|Q_{min}| = |M_{max}|$ then

Q_{min} must be optimal

define: assume we have min cover Q

$$\underset{\substack{\text{cover verts}\\\text{in } X}}{R = Q \cap X} \qquad \underset{\substack{\text{cover verts}\\\text{in } Y}}{T = Q \cap Y}$$

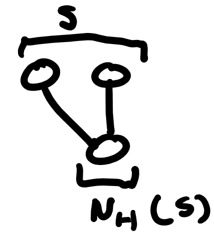
H = subgraph induced on $R \cup (Y - T)$

H' = subgraph induced on $T \cup (X - R)$

As $R \cup T$ is a vertex together

↳ no edges between $(Y-T), (X-R)$

$\hookrightarrow$ no edges between $(Y-T),(X-R)$



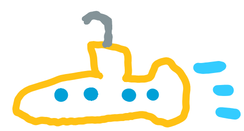
Note: if $\exists\, S \subseteq R : |N_H(S)| < |S|$

in H subgraph

$\longrightarrow Q - S + N_H(S)$ is a smaller vertex cover contradiction

So per Hall: R has a saturating match on H

Using same logic: T has a saturating match on H'

all together now: 

Min cover Q of size $|Q|$

$|Q| \geq |M|$ ← all possible matches

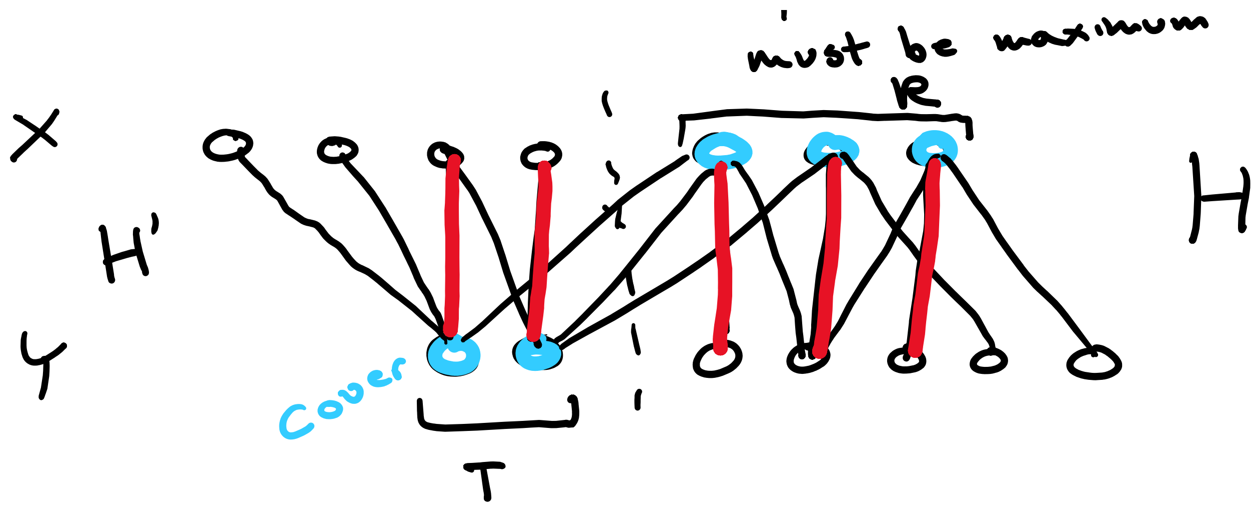
match M of size $|R| + |T| = |M|$

$\hookrightarrow$ and $|R| + |T| = |Q|$

$\Rightarrow |Q| = |M| \quad \square$

↑ must be maximum

R



This is a good example of a min-max relation aka a dual optimization problem

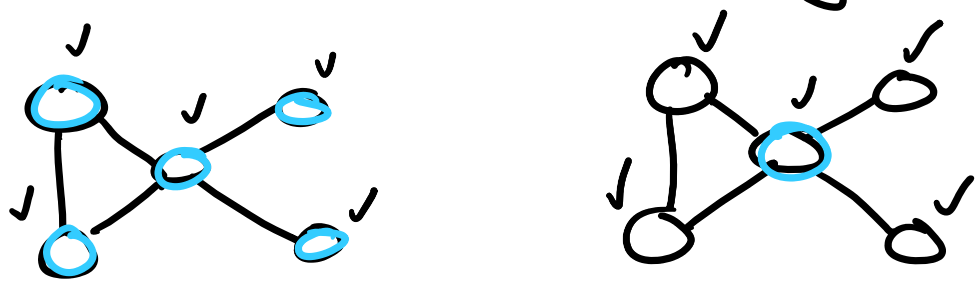
↳ solution to a minimization problem is equal to the solution for the maximization problem and vice-versa (sp?)

Dominating sets

<u>Dominating set</u>: $D \subseteq V(G)$ such that $\forall v \in V(G), v \notin D$

$\rightarrow \exists u \in N(v) : u \in D$

aka
all vertices are in or adjacent to a dominating set



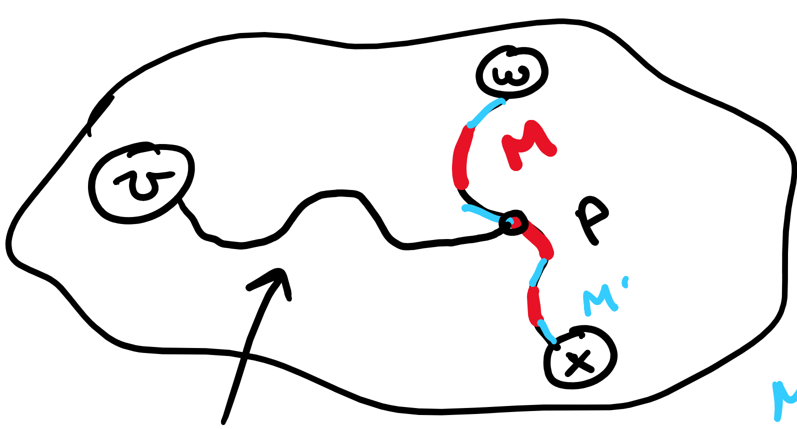
trivially, V(G) is a D.S.

minimum D.S.

Independent dominating sets

dominating sets using independent vertices

HW3 P1



can assume $|M| < |M_{max}|$

$M' = P \Delta M$

How can this

How can this path be configured w.r.t. P, M, and M'?