

10.1 Matching in General Graphs

For the most part, we've discussed matching restricted to bipartite graphs. We're now going to generalize some of the concepts we've discussed to all graphs. First define the function $o(G)$ as the number of **odd connected components** in G . As we know, an odd connected component has an odd number of vertices. In order to characterize which graphs have a perfect matching, we will use what could loosely be considered as a generalization of Hall's Condition.

Tutte's Theorem states that a graph G with a perfect match satisfies the inequality $\forall S \subseteq V(G) : o(G - S) \leq |S|$. Formally, a graph $G = (V, E)$ has a perfect matching if and only if for every possible vertex set $S \subseteq V(G)$, the subgraph induced by $V - S$ has at most $|S|$ connected components with an odd number of vertices. Let's develop a proof for Tutte's Theorem, while noting that Hall's Theorem is simply a restricted version of Tutte's Theorem.

10.2 Independent Sets, Dominating Sets, and Covers

A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint on all $e \in E(G)$. The vertices in Q *cover* $E(G)$. An **edge cover** of G is a set $L \subseteq E(G)$ such that L has at least one edge incident on all $v \in V(G)$. The edges in L *cover* $V(G)$.

The **König-Egerváry Theorem** states that if G is a bipartite graph, then the size of a maximum matching in G equals the minimum size of a vertex cover. We can prove this theorem too.

An **independent set** of vertices on a graph G are a set of vertices that are not connected by an edge. The size of a maximum independent set on G is called the **independence number** of G . For a bipartite graph, this isn't necessarily the size of the larger partite set.

In G , $S \subseteq V(G)$ is an independent set if and only if \bar{S} is a vertex cover. Thus a maximum independent set is the complement of a minimum vertex cover, and their sizes summed equals the order of G . You'll be proving this in the third homework.

Similar to a vertex cover is the notion of **dominating sets**. A dominating set is some vertex set $S \subseteq V(G)$ where $\forall v \in V(G) : v \in S$ or $\exists u \in N(v)$ such that $u \in S$. That is, a set S is dominating on some G if all vertices of G are either in S or have a neighbor in S .