Lecture 11 - Connectivity

Monday, March 9, 2026 8:40 AM

A human being should be able to change a diaper, plan an invasion, butcher a hog, conn a ship, design a building, write a sonnet, balance accounts, build a wall, set a bone, comfort the dying, take orders, give orders, cooperate, act alone, solve equations, analyze a new problem, pitch manure, program a computer, cook a tasty meal, fight efficiently, die gallantly. Specialization is for insects.

- Robert A. Heinlein

However:

- Financial/career success, fame, and similar achievements is generally a result of extremely high specialization

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One firm rule

↳ No HWs accepted after solutions posted

Review of connectivity

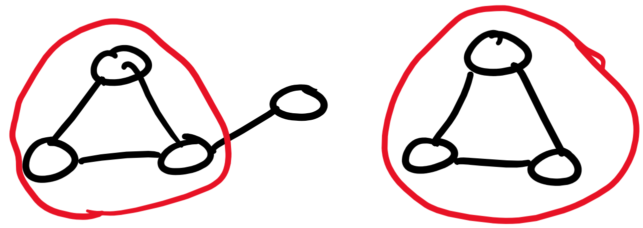
G is connected if

$\forall u, v \in V(G): \exists\, u,v$-path

connected component

maximal connected subgraph

G



component

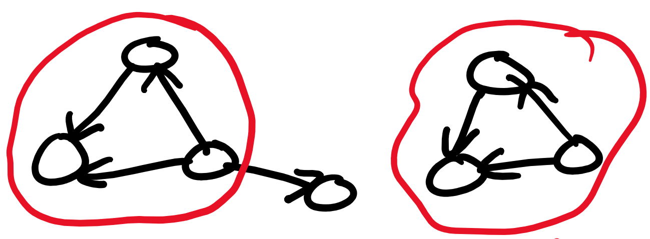
not a component

weak connectivity of digraphs

A digraph is <u>weakly connected</u> if the <u>underlying graph</u> is connected

↳ undirected graph if we remove edge directivity

D



not weak component

weak component

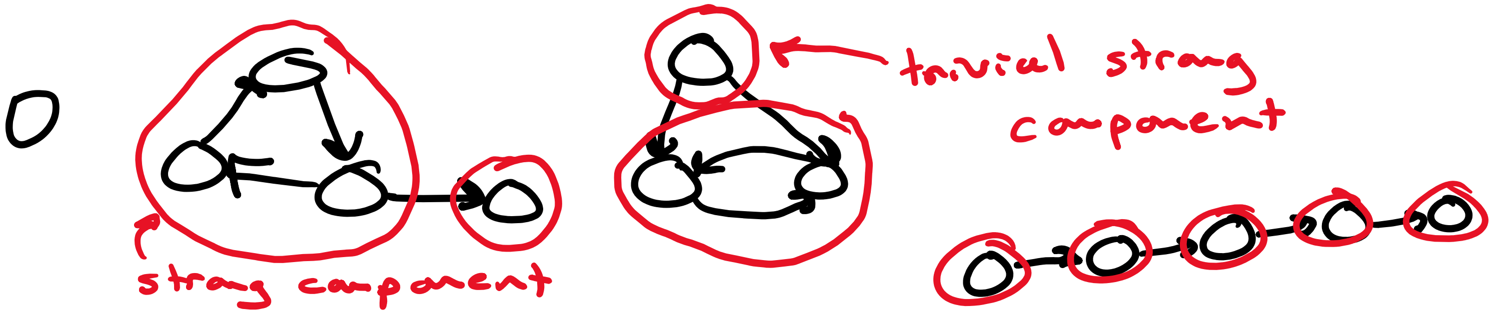
<u>weak component</u> is a maximal weakly connected subgraph

strong connectivity

A digraph is <u>strongly connected</u>

if $u,v \in V(D)$: $\exists$ u,v-path

(directed path)



A <u>strong component</u> is a maximal strongly connected subgraph

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Vertex Connectivity

(undirected graphs)

Recall: <u>cut vertex</u> is a vertex s.t. $G-v$ has more components than G

<u>Separating set:</u> a set $S \subseteq V(G)$ s.t. $G-S$ has more components than G

than G

(or is a single vertex)

AKA: vertex cut ← (sp?) preferred term

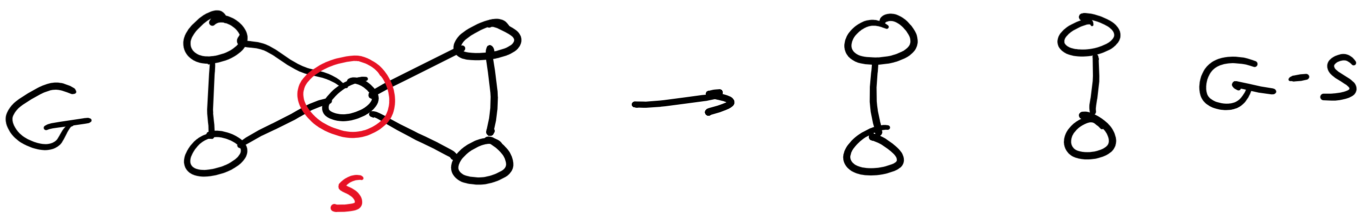
vertex separator

<u>Connectivity of G</u>: $\kappa(G) = k$

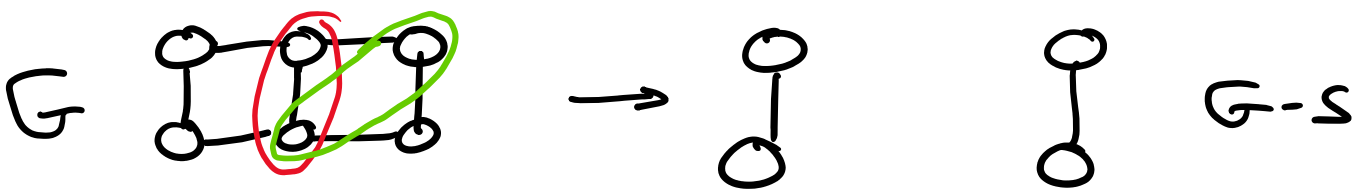
is the size of a minimum vertex cut

<u>G is k-connected</u> (k-vertex-connected)

if $\kappa(G) = k$



G is 1-connected



G is 2-connected

(Assume $|V(G)| \geq 2$)

Note: for connectivity, the maximum size of a vertex cut is $|V(G)| - 1$

↳ K_n is $(n-1)$-connected

underlying assumption

Assume G is at least connected

Examples: Tree T is 1-connected

star S_n is 1-connected

Cycle C_n is 2-connected

Edge Connectivity

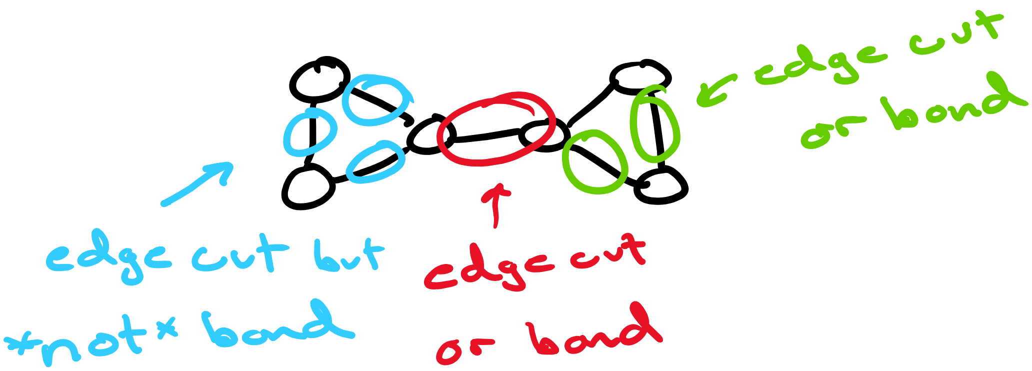
<u>cut edge</u>: an edge in some G s.t. $G - e$ has more components than G

__Disconnecting set:__ a set of edges $F \subseteq E(G)$ s.t. $G - F$ has more components than G

AKA: edge cut ← preferred term (sp?)

edge separators

__Bond:__ minimal edge cut



If F is a bond $\Rightarrow G - F$ has 2 components

If we disconnect G into > 2 components, that implies that there is a smaller subset of F that disconnects G into 2 components exactly

2 components exactly

<u>edge connectivity</u> of G: $\kappa'(G) = k$

minimum size of an edge cut

↳ G is k-edge-connected if $\kappa'(G) = k$

Examples: K_n is $(n-1)$-edge-connected

Tree T is 1-edge-connected

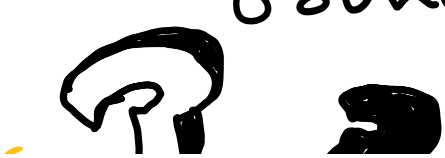
cycle C_n is 2-edge-connected

star S_n is 1-edge-connected

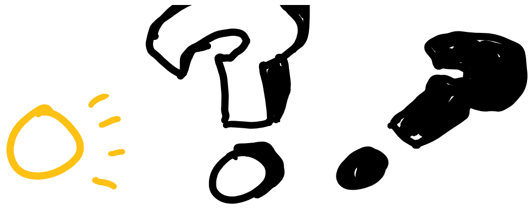
One final terminology consideration

→ If G k-connected, then G is also at least $(k-1)$-connected, $(k-2)$-connected, etc.

Bounds on connectivity



$$\kappa(G) \overset{?}{\leq} \kappa'(G) \overset{?}{\leq} \delta(G)$$



$$\kappa(G) \overset{?}{\lesseqgtr} \kappa'(G) \overset{?}{\lesseqgtr} \delta(G)$$

Q: can we place bounds between these values relative to each other?

→ Trivially, removing all edges of neighbors of some minimum degree vertex in G will always disconnect G

$$\kappa(G) \leq \delta(G)$$

$$\kappa'(G) \leq \delta(G)$$

But how are $\kappa(G)$ and $\kappa'(G)$ bounded relative to each other?

- Consider a minimum edge cut F that separates $V(G)$ into $S, \bar{S}$ s.t. $\bar{S} = V(G) - S$

(Note: $F = [S, \bar{S}]$)

Case 1: $\forall u \in S, \forall v \in \bar{S} : \exists (u,v) \in E(G)$

this implies

$$K'(G) = |F| = |S||\bar{S}| \geq |V(G)| - 1 \geq K(G)$$

$$\hookrightarrow K'(G) \geq K(G)$$

Case 2: $\exists x \in S, \exists y \in \bar{S} : (x,y) \notin E(G)$

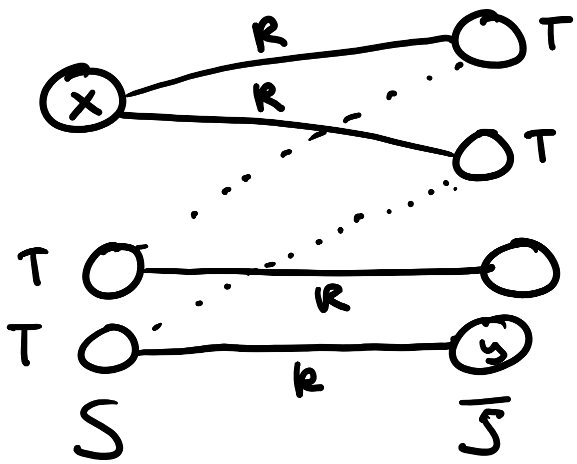
define $T = \{$all $u \in N(x) : u \in \bar{S}$ and
all $v \in S - x : \exists (v,z) \in E(G)$
$z \in \bar{S}\}$

Note: all x,y-paths must go through
some $w \in T$

(aka T is a vertex cut)

define $R = \{$all $e \in (x,w) : w \in T \cap \bar{S}$
and $f \in (a,b) : a \in T \cap S, b \in \bar{S}$
(with only one edge selected
for each a) $\}$

(aka R is ~~a edge cut~~)
at most as big as an edge cut)



Note: $|R| = |T|$

As we've only selected one f for each possible R out of many possible edges

$|R| \leq |F|$

$|T| \leq |R| \leq |F|$

$\hookrightarrow \kappa(G) \leq \kappa'(G)$

=> all together now ♫ ♩ ♩

$$\boxed{\kappa(G) \leq \kappa'(G) \leq \delta(G)}$$