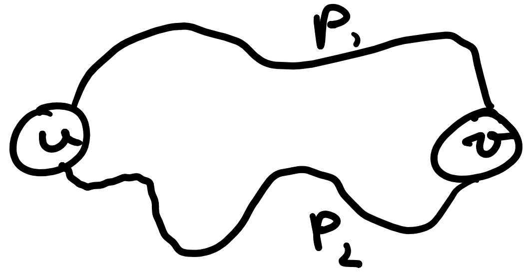
2-connectivity

→ must remove 2 vertices to disconnect G

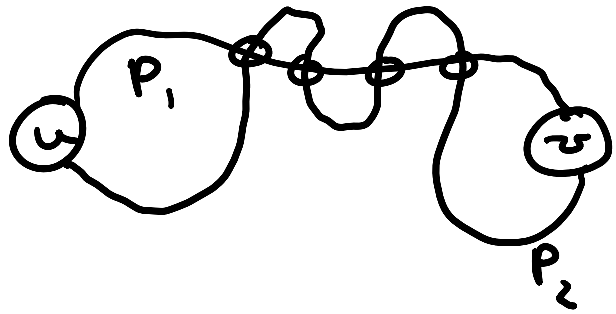
Internally disjoint paths

Paths between some u, v that share no internal vertices



Internally edge-disjoint paths

Paths between u, v that share no internal edges



Whitney's Theorem

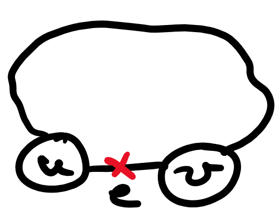
G where $|V(G)| \geq 3$ is at least 2-connected iff $\forall u, v \in V(G)$:

$\exists P_1 P_2$ u,v-idps

internally disjoint paths

($\Rightarrow$) Induction on $d(u,v)$

distance

Basis $P(1)$: 

$P_1 = \{e\}$

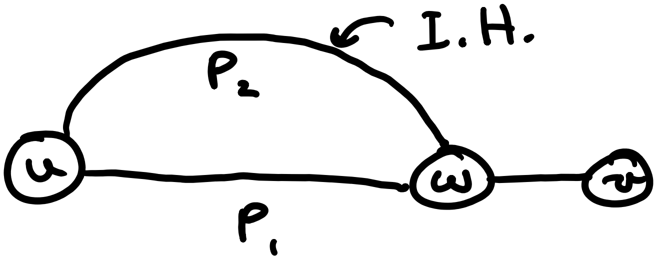
P_2 = any path on $G-e$

$K(G) \leq K'(G)$

Consider G with $d(u,v) = n$

$\rightarrow \exists$ at least one u,v-path

on this path, consider $w \in N(v)$



$d(u,w) = n-1 = k$

$\hookrightarrow$ I.H. gives us $P_1\ P_2$ u,w-idps

Note: as G is 2-connected

$\hookrightarrow \exists P_3 = u,v$-path on $G-w$

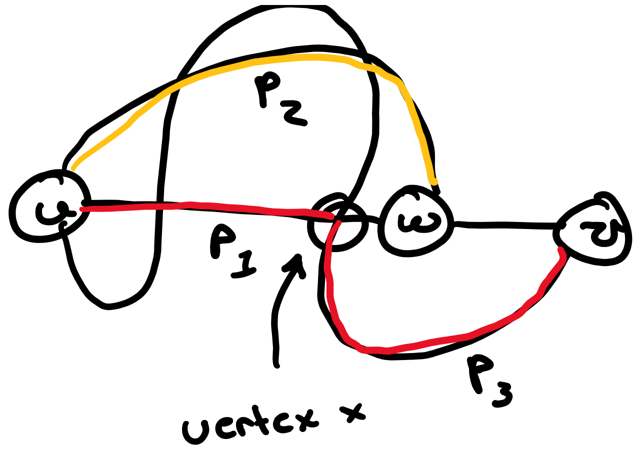
Q: Is P_3 internally disjoint to $P_1\ P_2$?

Case 1: yes $\rightarrow P = P_1 + (w,v)$

other $P = P_3$

Case 2: no :($\rightarrow P_3$ intersects P_1 and/or P_2 some number of times

number of times

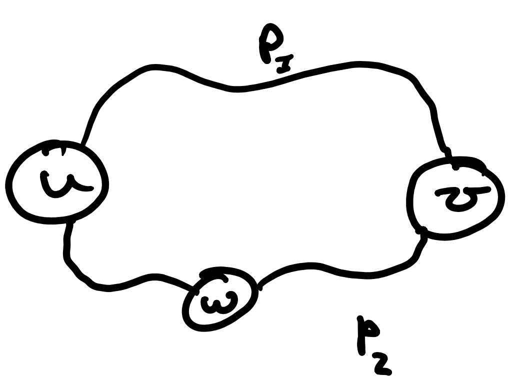


Define: x as the last vertex on P_3 that intersects with P_1 or P_2 (wlog say x is on P_1)

$\Rightarrow$ first $P = P_2$

second $P = (P_1$ from u to $x) +$ $(P_3$ from x to $v)$ ✓

$(\Leftarrow)$ Consider any $u, v \in V(G)$ and their 2 u,v-idps



Consider any w along P_1 or P_2

$\rightarrow$ u and v are still connected on $G-w$

$\Rightarrow$ minimum vertex separator must be at least 2 vertices $\square$

2-connected G

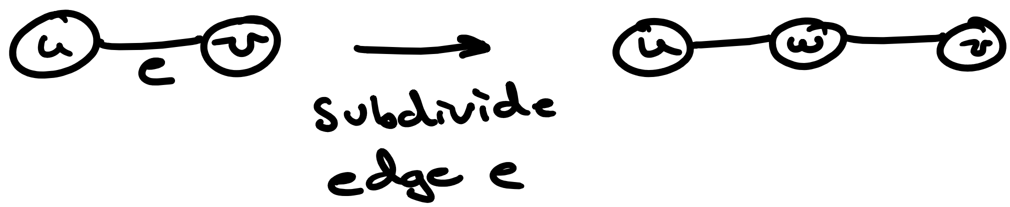
$\iff \exists P_1 P_2$ idps $\forall u, v \in V(G)$

$\iff \forall u, v \in V(G) : \exists C$ where $u, v \in V(C)$

$\iff \forall e, f \in E(G) : \exists C$ where $e, f \in E(C)$

To show the last statement, we'll use the concept of

Subdivisions



Note: subdivision preserves the 2-connectedness of G

→ so subdividing e, f to w, w' preserve our 2-connectedness and therefore 2-idps from w to w'

$\Rightarrow \exists\, C$ with $w, w' \in V(C)$

and equivalenty that cycle would have e, f in original G $\square$

Ear Decompositions

Recall: decomposition of G is a listing of subgraphs of G such that all edges of G are each in exactly one subgraph in the list

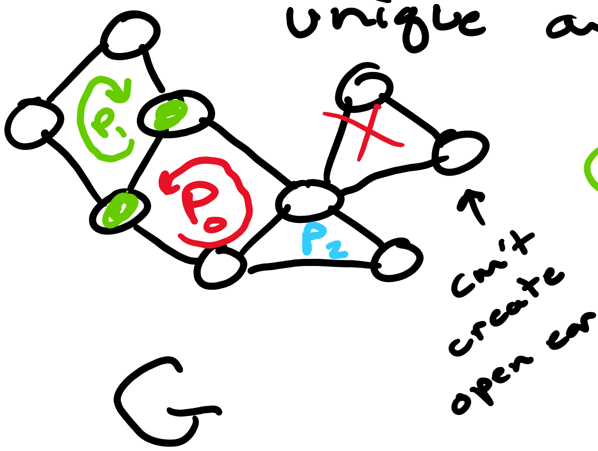
Open-ear decomposition

A decomposition of some G s.t.

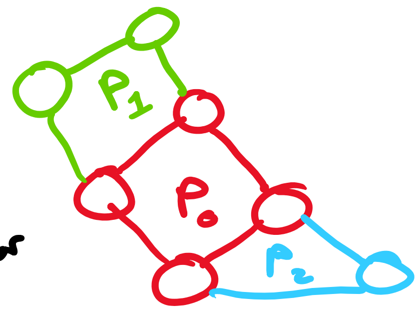
$D = P_0\, P_1\, P_2 \dots P_k$ ← subgraphs in decomposition

P_0 = cycle

P_i = a path whose endpoints are unique and exist on $P_0 P_1 \ldots P_{i-1}$



G



Decomp. of G

Q: Is the existence of an open ear decomposition necessary & sufficient for 2-connectedness?

$\exists$ open ear decomp $\Leftrightarrow$ 2-connected

($\Leftarrow$) Proof by ALGORITHM

select any $u, v \in V(G)$

$\hookrightarrow \exists C$ s.t. $u, v \in V(C)$

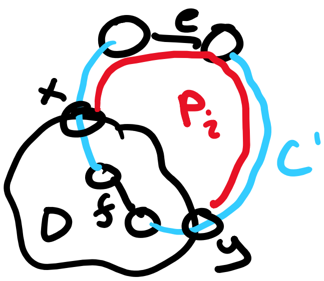
$P_0 = C$

while $\exists e \in E(G)$ s.t. $e \notin P_0 \ldots P_{i-1}$

while $\exists e \in E(G)$ s.t. $e \notin P_0 \ldots P_{i-1}$

Consider any $f \in P_0 \ldots P_{i-1}$

$\hookrightarrow \exists C'$ s.t. $e, f \in E(C')$



P_i = e follows C' in both directions until some x, y vertices in $P_0 \ldots P_{i-1}$ are reached

$\Rightarrow$ this builds our decomposition ✓

($\Rightarrow$) To show: prove adding an open ear to a 2-connected graph retains 2-connectedness

First consider P_0

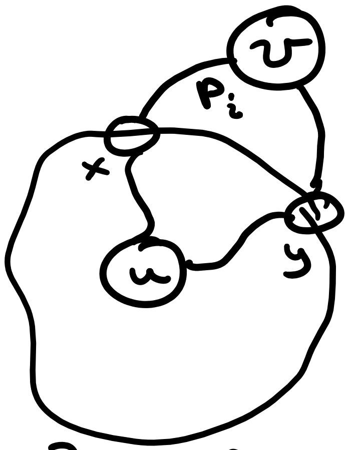
$\hookrightarrow$ cycles are 2-connected

Next, consider some P_i open ear

consider some $v \in P_i$

consider some $u \in P_0 \ldots P_{i-1}$

Q: can we guarantee 2 u,v-idps?

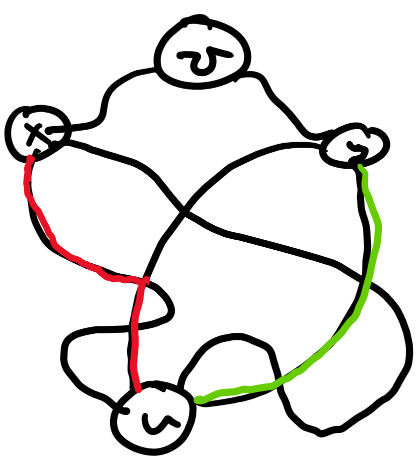


$D = P_0 \ldots P_{i-1}$

first $P = (v$ to x along $P_i) +$
$(x$ to u within $P_0 \ldots P_{i-1})$

second $P = (v$ to y along $P_i) +$
$(y$ to u within $P_0 \ldots P_{i-1})$

Q2: can we guarantee that our x,u and y,u-paths are internally disjoint?



A: yes, using same logic as with our first proof

$\Rightarrow$ Adding an open ear to a 2-connected graph will retain 2-connectedness, so any graph with an open ear decomposition will be 2-connected $\square$

2-edge-connectivity

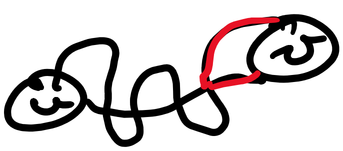
G is 2-edge-connected if
the size of a minimum
edge cut is 2 edges

↳ similar ideas to 2-connectivity
in that we guarantee the existence
of a closed trail containing
any pair of vertices (or edges)

G is 2-edge-connected $\iff$

$\forall u, v \in V(G): \exists\, 2$ u,v-edps

internally edge disjoint paths



to prove, we can use the
same logic for our 2-connected
proof to find our 2 edps

G is 2-edge-connected $\iff$
G has a closed ear decomposition

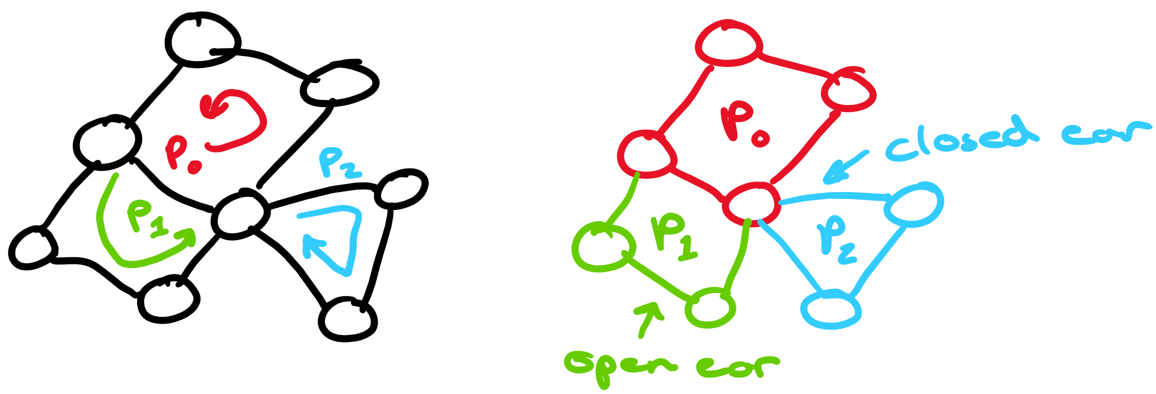
<u>Closed ear decomposition</u>

Closed ear decomposition

P_0 = cycle

P_i = open OR closed ear whose endpoint(s) is (are) on $P_0 \dots P_{i-1}$

a closed path aka a cycle



G is 2-edge-connected

$\Leftrightarrow \forall u,v \in V(G): \exists\, 2$ u,v-edps

$\Leftrightarrow \forall e,f \in E(G): \exists$ closed T (trail) where $e,f \in E(T)$

$\Leftrightarrow G$ has a closed ear decomposition

Biconnectivity

→ a biconnected graph G where $|V(G)| \geq 3$ is 2-connected

OR is K_1 or K_2

o   o—o

Block decomposition of G

<u>blocks</u>: maximal biconnected components of G

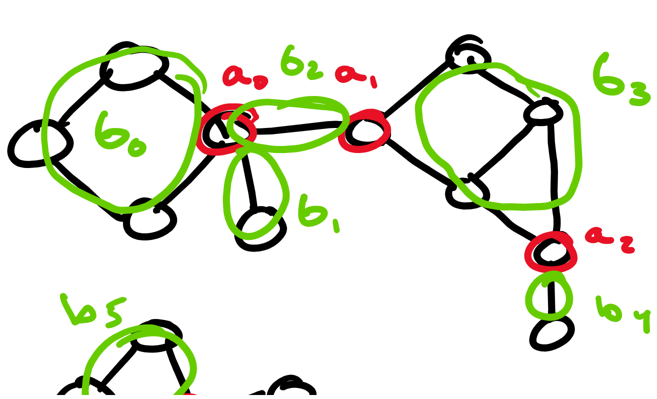
<u>articulation vertices</u>: cut vertices

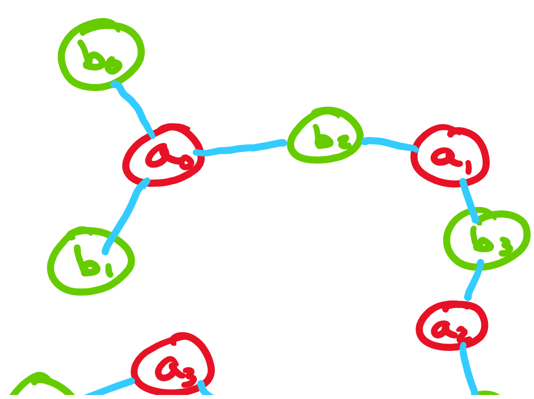
<u>bridges</u>: cut edges

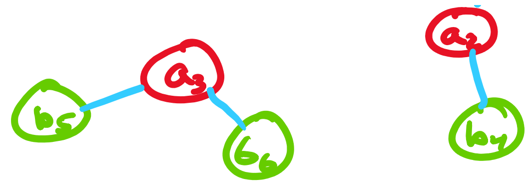
Using the above, we can construct a <u>block-cutpoint</u> graph G_B

$V(G_B) = \{$ blocks & articulation vertices $\}$

$E(G_B) = \{$ pairwise block memberships for all articulation vertices $\}$







G $\qquad\qquad$ G_B

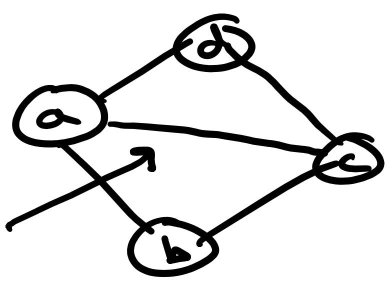
Note: G_B is a forest

leaves are BiCCs

↑

biconnected components

Weekly Problem 6

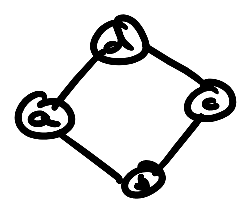


induced subgraph on $\{a, b, c, d\}$ would be everything shown

chord intersects a cycle

Show: can't exist in minimally 2-connected graphs

non-induced cycle subgraph



induced: we use all edges between all vertices

between all vertices
in our subgraph

non-induced: we can
select a subset of edges