Lecture 13 - Digraph and k-connectivity

Thursday, March 12, 2026 3:36 PM

Best song lyric?

- There are ten million million million million million million million million million million particles in the universe that we can observe, your mama took the ugly ones and put them into one nerd.
- Stephen Hawking vs. Albert Einstein, Epic Rap Battles of History

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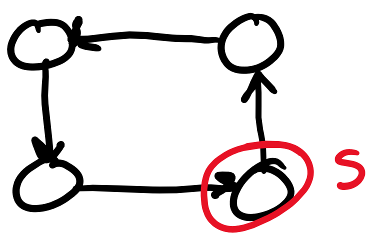
Digraph connectivity

<u>vertex cut</u> – a set $S \subseteq V(D)$ s.t. $D - S$ is not strongly connected



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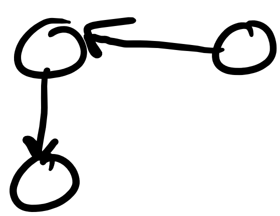


← not strongly connected

D−S

$\kappa(D)$ = connectivity of D

$= \min |S|$ → aka size of minimum cut

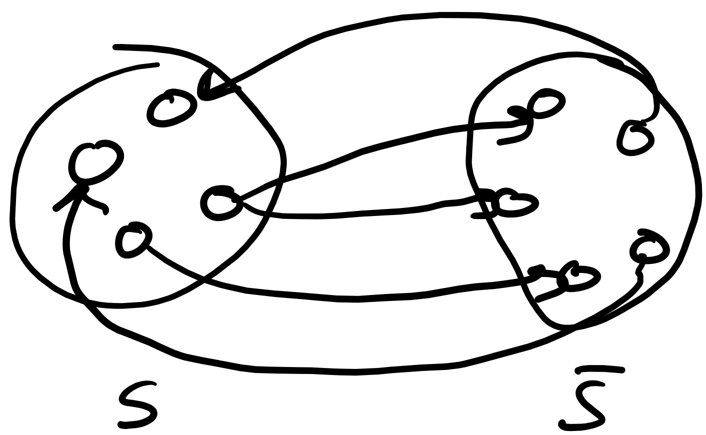
$\forall S \subseteq V(D)$

<u>edge cut</u> – a set $F \subseteq E(D)$ that separates $V(D)$ into two vertex sets $S, \bar{S}$

two vertex sets $S, \bar{S}$

$\bar{S} = V(D) - S$

The size of some cut F is the number of edges from $S \to \bar{S}$



$|[S, \bar{S}]| = 3$

size of cut

$|[S, \bar{S}]| \neq |[\bar{S}, S]|$

not necessarily equal

$K'(D)$ = edge-connectivity of D

$= \min_{S \subseteq V(D)} |[S, \bar{S}]|$

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k-connectivity (undirected graphs)

<u>x, y-separator</u> - a set $S \subseteq V(G)$ s.t. $G - S$ has no x, y-path

$K(G)$: minimum x, y-separator over all possible $x, y \in V(G)$

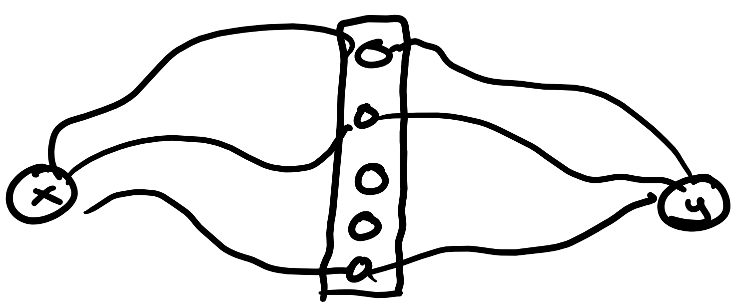
over all possible $x, y \in V(G)$

$K(x,y)$ = connectivity of x, y
aka size of a minimum x,y-separator

$\lambda(x,y)$ = maximum number of x,y-idps

First note: every x,y-separator must contain a vertex from each x,y-idp

$\Rightarrow K(x,y) \geq \lambda(x,y)$



S

? BIG ? ?

? QUESTION

does $\lambda(x,y) \stackrel{?}{=} K(x,y)$

Whitney: for $2 = \lambda(x,y) = K(x,y)$
it holds ✓

Menger: it also holds for
any $k = \lambda(x,y) = K(x,y)$
$\rightarrow$ if $(x,y) \notin E(G)$

Let's prove this using
the **power** of strong induction

Induction on $|V(G)|$

Basis $P(2)$: x y $\lambda(x,y) = 0$
$K(x,y) = 0$
$\lambda = K$ ✓

Assume we have some G
s.t. $|V(G)| = n$

Also assume for some x, y we have

Also assume for some x, y we have

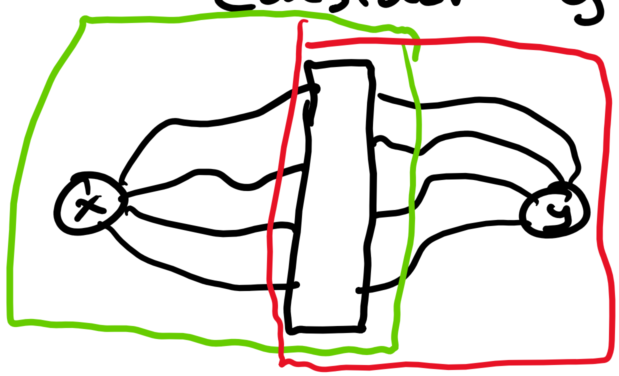
$K(x,y) = k = |S|$ ← min separator

Our goal: construct k idps given S

Case 1: $\exists S$ s.t. $S \neq N(x)$, $S \neq N(y)$

consider x, S-paths

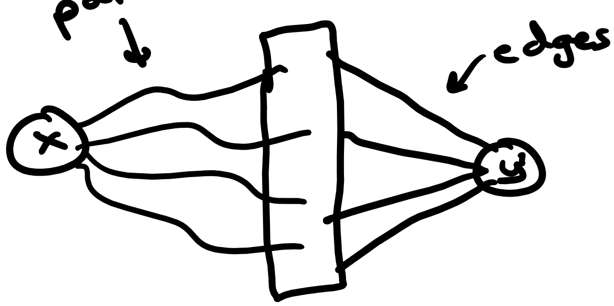
consider y, S-paths



define graph

$H_1 = H_x + y'$

where we have edges from each S to y'



I.H. on H_1 gives us k idps from x to y'

define $H_2 = H_y + x'$

where we have edges from each S to x'

I.H. on H_2 gives us k x,y-idps

↳ combine the two sets of paths to give us k x,y-idps on our original G ✓

Case 2: $S = N(x)$ or $S = N(y)$

2a) $\exists v \notin \{x\} \cup \{y\} \cup N(x) \cup N(y)$

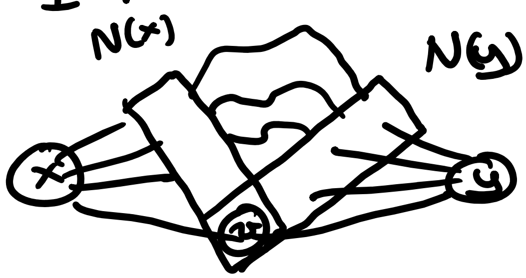
Note: v is not on a min cut

I.H. on $G-v$ gives us k x,y-idps

↳ these are the same as on G ✓

2b) $\exists v \in N(x) \cap N(y)$

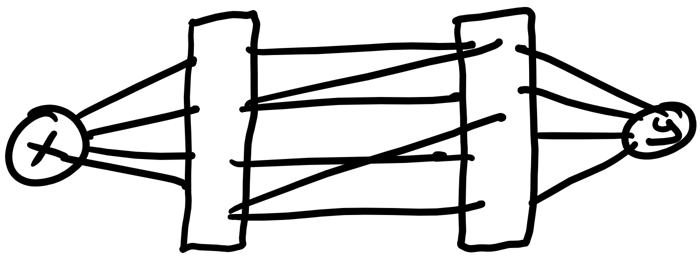
→ I.H. on $G-v$ gives us $k-1$ x,y-idps



↳ to get k idps on G we also have path $\{(x,v),(v,y)\}$ ✓

2c) We have both $N(x)$ and $N(y)$ as

(c) we have both N(x) and N(y) as minimum separators with no "external" vertices



only possible configuration

N(x) N(y)

Note: this is a bipartite graph

Note 2: every x,y-path uses some edge in this bipartite graph

↳ we can construct idps by creating a match on this bipartite graph

$|\text{max match}| = \text{max \# of } x,y\text{-idps}$

Q: Can we guarantee k-idps on a match of size k on this bipartite graph $G_{N(x),N(y)}$?

Note 3: each of $N(x)$ and $N(y)$ are minimum covers on $G_{N(x),N(y)}$

as $k = |N(x)| = |N(y)| = |\text{max match}|$

$\Rightarrow$ we can construct k x,y-idps on the original G $\square$

$$\Rightarrow K(x,y) = \lambda(x,y)$$

and over all possible selections of x,y we can state that if G has connectivity k that implies $\forall x,y \in V(G): \exists\, k$ x,y-idps ✓

what about edge connectivity?

$K'(x,y)$ = minimum size of an x,y-disconnecting set

$\lambda'(x,y)$ = maximum number of edps between x and y

edps: edge disjoint paths

paths

AND $\kappa'(x,y) = \lambda'(x,y)$

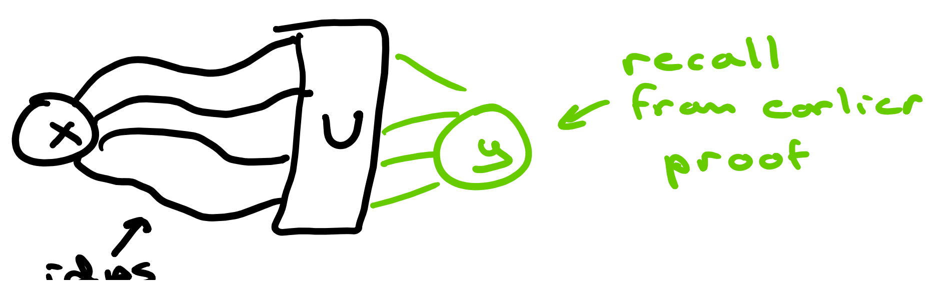
Proof: exercise for reader (similar approach)

G is k-connected if
$\forall x,y \in V(G)$: $\kappa(x,y) \geq k$
$\lambda(x,y) \geq k$

G is k-edge-connected if
$\forall x,y \in V(G)$: $\kappa'(x,y) \geq k$
$\lambda'(x,y) \geq k$

Let's get generalizing

<u>x, U-fan</u> : a set of paths from x to vertex set U s.t. the paths only share vertex x (aka idps)



G is k-connected iff $|V(G)| \geq k+1$
and $\forall x \in V(G)$, $U \subseteq V(G)$, $|U| \geq k$:
$\exists$ x, U-fan of at least size k

($\Rightarrow$) Construct G' as $G+y$, where
y is attached to all $u \in U$

Note: G' is also k-connected, as
the addition of y cannot
create a smaller vertex
cut on G

$\rightarrow$ Per Menger, we have k x,y-idps
and therefore a x, U-fan of size k ✓

($\Leftarrow$) $\forall v \in V(G)$ and $U = V(G) - v$
we have v, U-fan of size k
per our assumptions
$\hookrightarrow \delta(G) \geq k$

Consider some $w, z \in V(G)$
and define $U = N(z)$

As $|U| \geq k$, we have a w,U-fan of cardinality k per our assumptions

→ the edges from U to z combined with the w,U-fan idps gives us k w,z-idps

=> Per Menger, G is k-connected □

We can use this result to prove another generalization of a result from Whitney

Recall: if G is 2-connected then
$\forall x,y \in V(G) : \exists C$ where $x,y \in V(C)$

In general: if G is k-connected then
$\forall S \subseteq V(G), |S| = k : \exists C$ where $S \subseteq V(C)$

Strong induction on k for some k-connected G

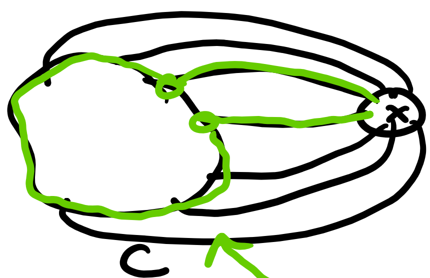
Basis $P(2)$ → we showed via Whitney last class

We have some k-connected G
and we have some $S \subseteq V(G) : |S| = k$

via I.H. $S - \{x\}$ has some $C : |V(C)| \geq k-1$
as the construction is $(k-1)$-connected

Case 1: $|V(C)| = k-1$

We know we have $x, V(C)$-fan
of size $k-1$



→ we have 2 idps
from x to 2
consecutive vertices
in C

→ we can expand C trivially
to include x

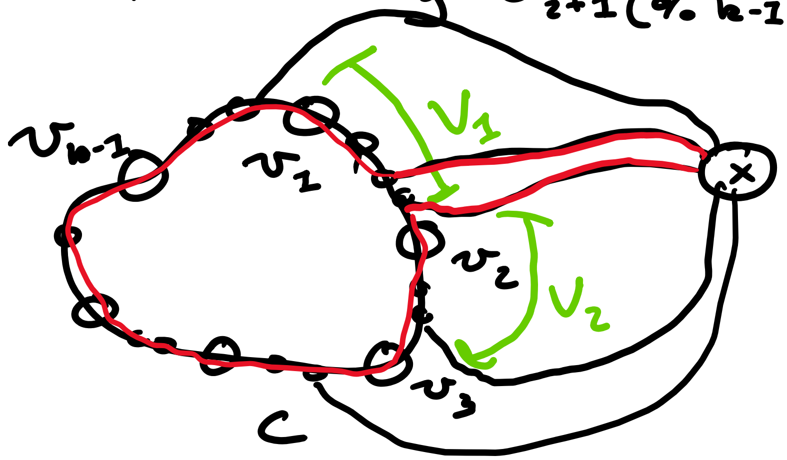
Case 2: $|V(C)| \geq k$

G has a $x, V(C)$-fan of size k

Order vertices of $S - \{x\}$ on C
as $v_1 v_2 \ldots v_{k-1}$

Order sets of all vertices on C
as V_i contains all vertices
from v_i up to but not

$\Rightarrow$ V_i contains all vertices from v_i up to but not including $v_{i+1(\% k-1)}$



$\rightarrow$ by pidgeonhole principle, at least two idps in the $x, V(C)$-fan must end on vertices in the same V_i subset

$\rightarrow$ We can modify our cycle to include x by detouring along 2 of these paths

$\Rightarrow$ we have a cycle containing all k vertices in our original S

$\square$