13.1 Digraph Connectivity

We can extend the concepts and terminology of connectivity to directed graphs as well. A **vertex cut** or **separating set** in a digraph D is a set $S \subseteq V(G)$ such that $D - S$ is not strongly connected. The **connectivity** $\kappa(D)$ is the minimum size of vertex set S such that $D - S$ is not strongly connected or is a single vertex. If $k \leq \kappa(D)$, then D is k-**connected**. A digraph is k-edge-connected if every **edge cut** has at least k edges, where an edge cut separates $V(D)$ into two sets $S, \bar{S}$ such that the size of the edge cut is the number of directed edges (u, v) from $v \in S$ to $u \in \bar{S}$. The **edge-connectivity** $\kappa'(D)$ is the minimum size of an edge cut. If $k \leq \kappa'(D)$, then D is k-**edge-connected**.

As we have noted, 2-edge-connected graphs share similarities with strongly connected digraphs. We can show that adding a directed ear to a strong digraph produces a larger strongly connected digraph.

13.2 k-Connected Graphs

We can now further extend a few of the concepts we discussed with restriction to 2-connected and 2-edge-connected to k-connected and k-edge-connected graphs. Given two vertices $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-**separator** if $G - S$ has no x, y-path. We define $\kappa(x, y)$ as the minimum cardinality over all possible x, y-separators and $\lambda(x, y)$ as the maximum cardinality over all possible sets of internally disjoint x, y-paths. Since any x, y-separator must contain an internal vertex of every internally disjoint x, y-path, we have $\kappa(x, y) \geq \lambda(x, y)$.

What follows is a generalization of Whitney's Theorem. **Menger's Theorem** states that for two vertices $x, y \in V(G)$ and $(x, y) \notin E(G)$ the minimum size of an x, y-separator equals the maximum number of pairwise internally disjoint x, y-paths; i.e, $\kappa(x, y) = \lambda(x, y)$. A graph is therefore k-**connected** if for all $x, y \in V(G)$, $\lambda(x, y) \geq k$.

We have similar concepts and terminology for k-edge-connectivity. Given two vertices $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-**disconnecting set** if $G - F$ has no x, y-path. We define $\kappa'(x, y)$ as the minimum cardinality over all possible x, y-disconnecting sets and $\lambda'(x, y)$ as the maximum cardinality over all possible sets of edge disjoint x, y-paths. A graph is k-**edge-connected** if for all $x, y \in V(G)$, $\lambda'(x, y) \geq k$. Likewise, $\kappa'(x, y) = \lambda'(x, y)$.

We can also generalize the notion of k-connectivity to cycles containing some set of k vertices, as an extension of our discussions with 2-connectivity. Let's prove that if G is k-connected, for any selection of $S \subseteq V(G)$ where $|S| = k$, there exists some cycle C where $S \subseteq V(C)$.