Best Star Treks:
1. Stargate SG-1
2. Galaxy Quest
3. The Orville
4. Deep Space 9
5. ....

(various movies/shows)

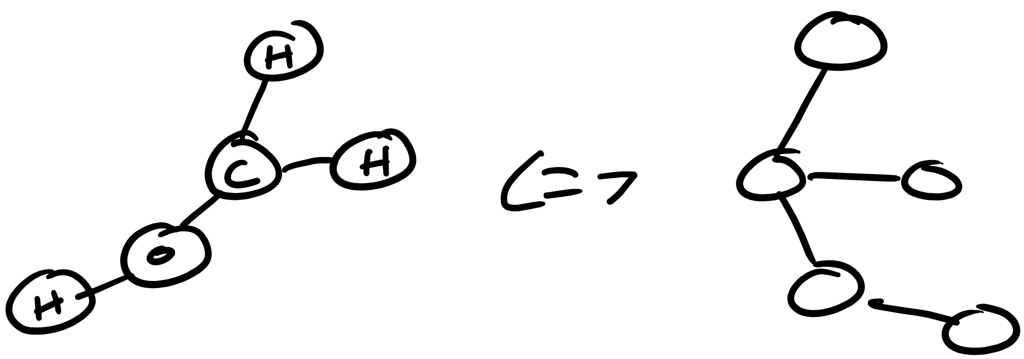
....

37. Original Series

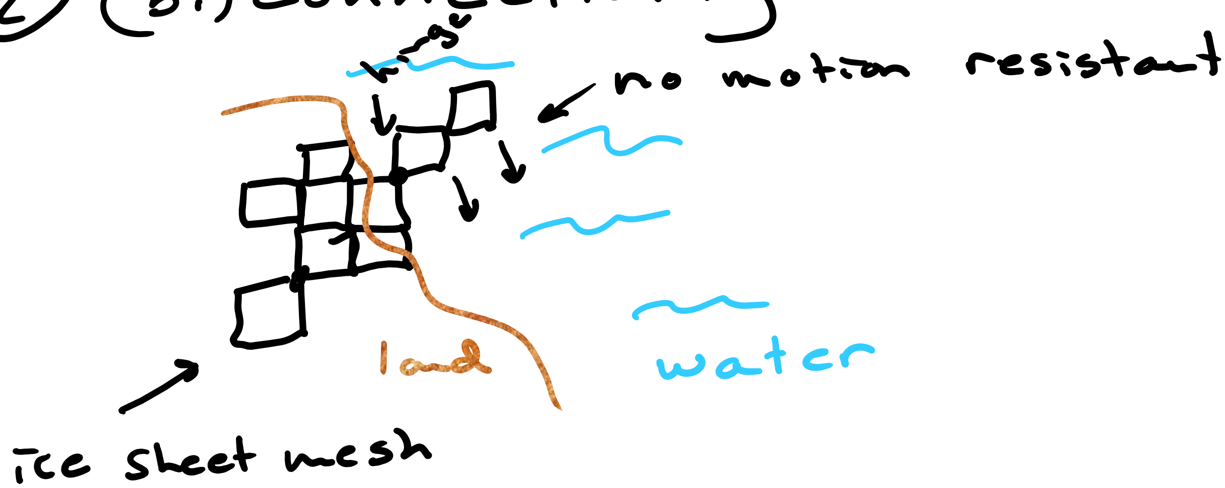
A Star Trek: sci-fi where characters explore space, find and interact with aliens that are pretty much human, all while making some social commentary.

Graph Applications

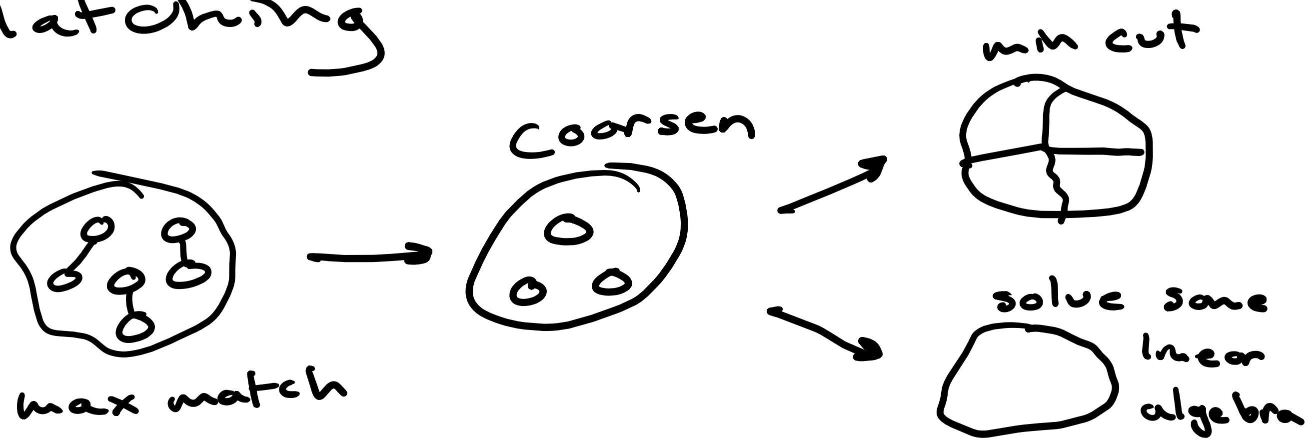
1 (sub)graph isomorphism



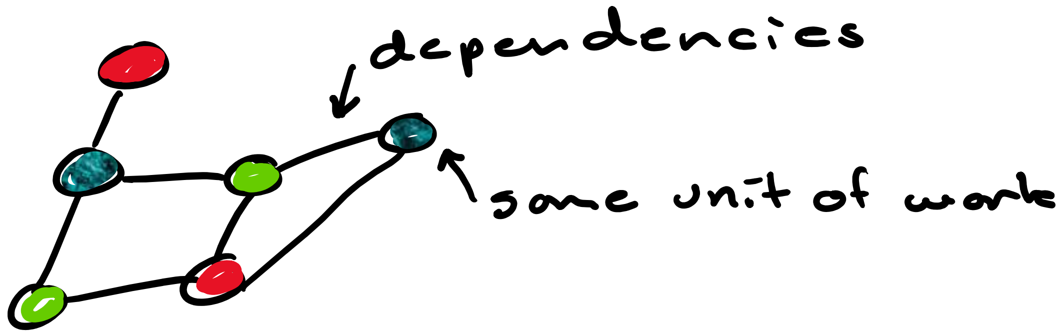
2 (bi)connectivity



③ Matching



④ Coloring



Bizniz → graph coloring (vertex coloring)

<u>k-coloring</u> of G is a labeling $f: V(G) \to S$, $k = |S|$ (set of colors)

<u>proper coloring</u> of G is a k-coloring s.t. no two neighboring vertices have the same color

<u>chromatic number</u> of $G = \chi(G)$

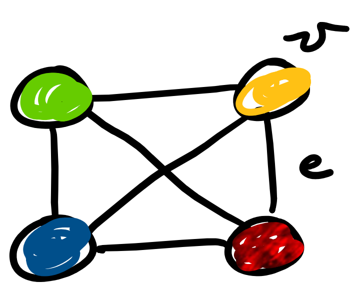
$\chi(G)$ is the minimum k for which G is properly k-colorable

<u>Optimal coloring</u> of G is a proper k-coloring for $k = \chi(G)$

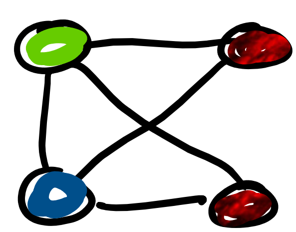
G is <u>color-critical</u> if for all subgraphs $H \subset G$, $H \neq G$
$\chi(H) < \chi(G)$

Note: $\forall e \in E(G)$: $\chi(G - e) < \chi(G)$

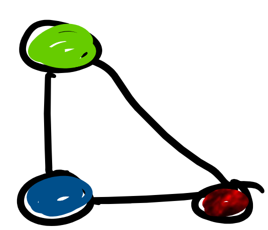
Note 2: all cliques are color-critical



K_4



$K_4 - e$



$K_4 - v$

$\chi(K_4) = 4$ $\chi(K_4 - e) = 3$ $\chi(K_4 - v) = 3$

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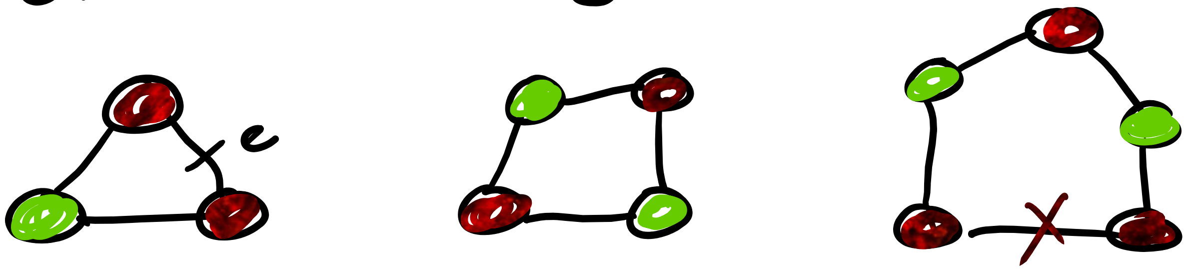
Note 3: odd cycles are color-critical



$K_3 \,/\, C_3$ C_4 $C_5 - e$

$\chi(\text{odd cycles}) = 3$

$\chi(\text{even cycle}) = 2$

IMPORTANT!!

↳ Each set of vertices of same color on a k-colored graph is an <u>independent set</u>

Greedy Coloring Algorithm

Initialize all $V(G)$ to "empty" color

For all $v \in V(G)$ in <u>some</u> order:

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color v with the "least" color not already in $N(v)$

Note: we usually consider colorset S as orderable

(require some notion of "least")

Greedy coloring will not always find an optimal solution

↳ But it can with some ordering of vertices

Proof: HW5 question

Overall: Quality of G.C. is dependent on the vertex processing order

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Let's talk bounds

Let's Talk bounds

(on G's chromatic number)

(assume G is simple)

In general for any G (non-null)

$$1 \leq \chi(G) \leq |V(G)|$$

For non-empty G

$$2 \leq \chi(G) \leq |V(G)|$$

If G is a tree / forest

$$2 = \chi(G)$$

In general, if G is bipartite

$$2 = \chi(G)$$

If G is a clique

$$\chi(K_n) = n$$

If G is an odd cycle

$$\chi(G) = 3$$

↳ even cycle $\chi(G)=2$

In general $\forall H \subseteq G$

$\chi(H) \leq \chi(G)$

Define $\omega(G)$ as the clique number

↳ $\omega(G)$ = the size of the largest clique in G

For all graphs

$\omega(G) \leq \chi(G)$

Consider our greedy algorithm

$\chi(G) \leq \Delta(G)+1$

Q: Can we improve this bound?

A: Per Brooks → yes we can

Brooks: $\chi(G) \leq \Delta(G)$

Brooks: $\chi(G) = \Delta(G)$

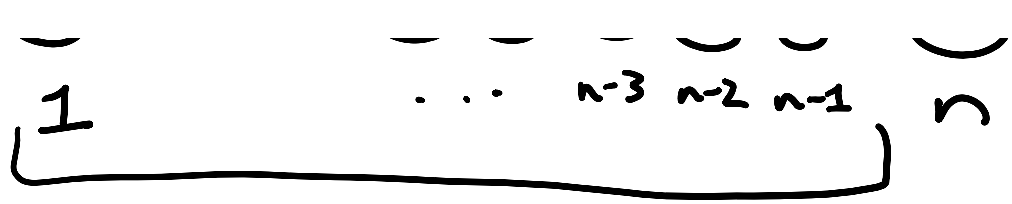
(except for odd cycles and cliques)

To prove: construct an ordering for greedy coloring s.t. we can guarantee each vertex has at most $\Delta(G)-1$ prior vertices processed

Case 1: G is not k-regular

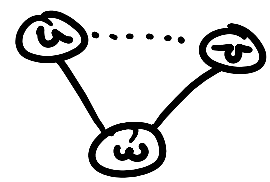
- select some $u \in V(G) : d(u) < \Delta(G)$
- grow a spanning tree from u
- apply order in reverse

→ every max degree is guaranteed at least one higher-ordered neighbor



any $\Delta(G)$ vertex is in here

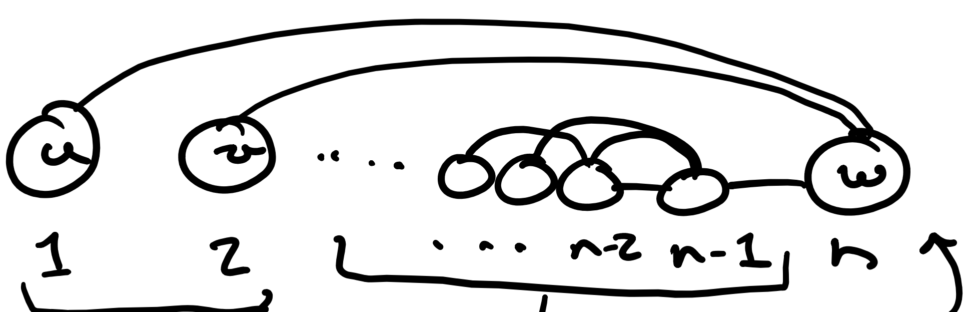
Case 2: G is k-regular



- there must be some $\{u, v\} \in N(w)$ s.t. $(u, v) \notin E(G)$

To construct our order

- u, v are listed first
- w is listed last
- grow a spaning tree from w and use the reverse order



$C(u) = C(v)$

same as above here

so w has at most $\Delta(G) - 1$ unique colors in $N(w)$

$\Rightarrow$ At most $\Delta(G)-1$ colors will show up in $N(x) : \forall x \in V(G)$ when processed by G.C. $\square$

Q: How tight are these bounds???

Consider: $\Delta(G), \omega(G), \chi(G)$

$$\omega(G) \overset{?}{\leq} \chi(G) \overset{?}{\leq} \Delta(G)$$

How tight

A: not very

For example:

Consider tree T

$\Delta(G) \to \infty$ (can be arbitrarily large)

$\chi(T) = 2$

$\chi(T) <<<<<<<<< \Delta(T)$

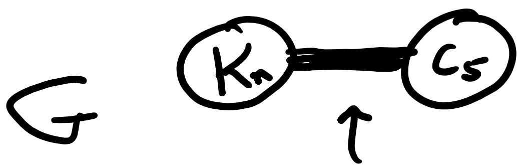
=> our upper bound can be arbitrarily loose in general

What about lower bounds?

$2 \leq \chi(G)$ if G is non-empty

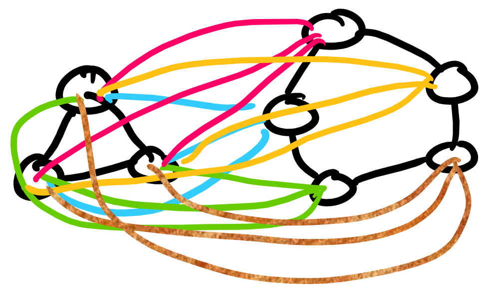
$\omega(G) \leq \chi(G)$ more generally

First: consider some graph where we can guarantee inequality in the above relation



graph join

→ all edges (u,v) exist $\forall u \in V(K_n)$
$\forall v \in V(C_5)$



Note: we need $n+3$ colors for G while $\omega(G) = n+2$

=> so $\omega(G)$ can be loose

Q: How looooose?

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A: $L\infty se$ ← infinitely

How can we show this?

→ we'll develop a construction that will increase $\chi(G)$ but will keep $\omega(G)$ fixed

AKA: Mycielski's Construction

Given a triangle-free G ($\omega(G)=2$) with $\chi(G)=k$, we will construct G' with $\chi(G')=k+1$ and $\omega(G')=2$

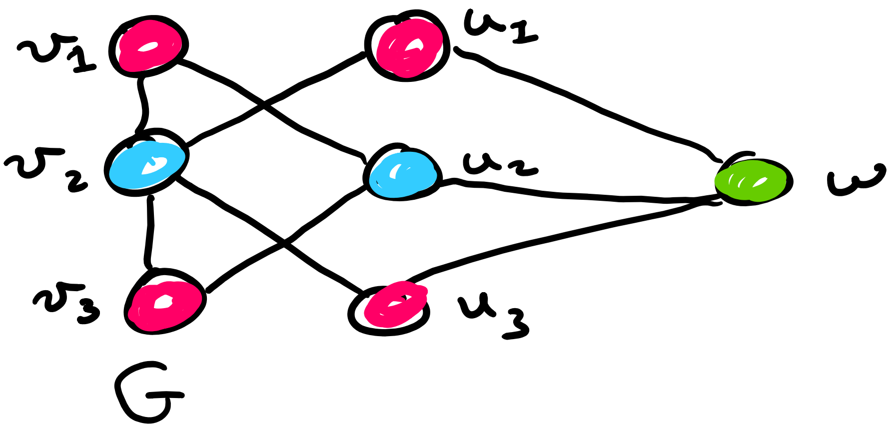
Consider: $\{v_1 v_2 \ldots v_n\} = V(G)$

create: $\{u_1 u_2 \ldots u_n\}$ in G'

add edges between u_i and all $v_j \in N(v_i)$

create vertex w and attach

create vertex w and attach it to all u_i vertices



$\chi(G) = 2$
$\omega(G) = 2$ $\xrightarrow{+1}$

$\chi(G') = 3$
$\omega(G') = 2$

Note: we don't create any triangles

Note 2: coloring all of u_i requires the same # colors for all of v_i

Note 3: w requires a "new" color

Note 4: we can iterate this construction infinite times

$$2 = \omega(G^{'''''' \times \infty}) <<<<<<< \chi(G^{'''''' \times \infty})$$

Takeaway: our bounds tell us very little in the general case

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Mostly due to the computational difficulty of the minimum coloring problem

Next time: minimum vertex coloring

→ computationally hard

→ so is getting $\chi(G)$