Lecture 16 - More Coloring

Thursday, March 26, 2026 8:59 AM

Did you know: you don't need to boil water before adding dry pasta

- And you actually probably shouldn't
- Fresh pasta you definitely need to, however

Dry pasta needs to do two things via environmental factors

1. Rehydrate -> does so at any temperature (higher=faster)
2. Cook -> does so only when the water is above 180F

Not worrying about water temperature when adding dried pasta confers several benefits:

1. Energy efficiency: you only need just enough water to cover the pasta
2. Major bonus: starch concentration in minimal water will be much higher
   a. How you get sauce to stick to pasta: starch from pasta water (can use less volume)
3. Time efficiency: min water and immediately adding pasta reduces overall cook time
   a. You should stir once or twice to prevent pasta from sticking

And detriments:

0. Literally none

Properties of k-chromatic graphs

P1: How small can a k-chromatic graph be? (edges)

→ Consider all color combinations

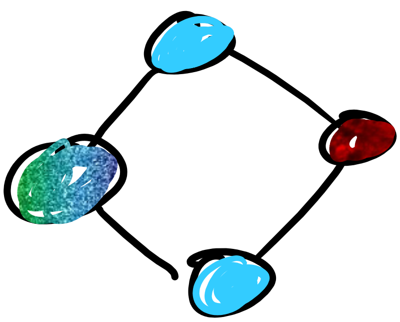
$\binom{k}{2}$ total number of color combinations across any given edge

→ Also going to be the minimum number of edges of a k-chromatic graph

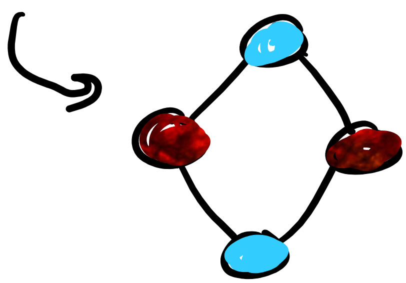
graph

whY?

Every combination of colors must exist on a k-coloring, otherwise we could just combine colorsets of a color combination to get a k-1 coloring



Possible color combos



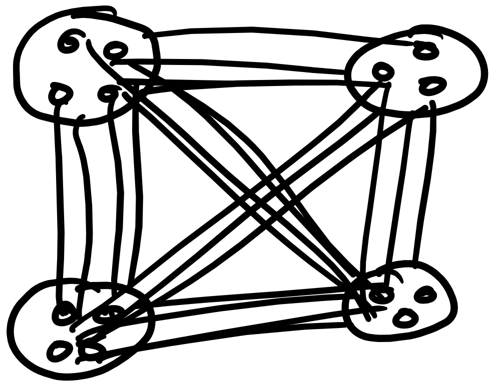
$\Rightarrow$ Any k-chromatic G must have $|E(G)| \geq \binom{k}{2}$ $\square$

What about how BIG?

Think in terms of a generalization of complete bipartite graphs

↳ complete <u>multi-partite</u> graphs

(k-partite)



← all edges exist between independent vertex sets

→ so a k-partite complete graph is largest k-colorable graph

Can we explicitly maximize $|E(G)|$ considering $k, |V(G)|$

→ Set all partite sets to be the same size +/- 1 vertex

→ Turán graph

Q: Does it maximize $|E(G)|$?

First, consider an unbalanced partitioning on a complete multi-partite graph

$\exists S_i S_j$ s.t. $|S_i| + 1 < |S_j|$

partite sets

move $v \in S_j$ to S_i

→ edges lost: $|S_i|$

→ edges gained: $|S_j| - 1$

→ as $|S_j| > |S_i| + 1$, we have a net gain on $|E(G)|$

Repeat this process, until eventually we reach a point where all sets are $^+/_-$ 1 vertex in size

↳ This maximizes our $|E(G)|$

$\Rightarrow$ Turán graph is the largest k-chromatic graph on $|V(G)|$ vertices $\square$

Color-critical graphs

If G is color-critical:

$$\forall v \in V(G):\ \chi(G-v) < \chi(G)$$
$$\forall e \in E(G):\ \chi(G-e) < \chi(G)$$

<u>k-color-critical</u> if $\chi(G)=k$

For k-color-critical G:

$\exists$ [illegible] G s.t.

$\exists$ some k-coloring on G s.t.
$\forall v \in V(G)$ the color $c(v)$
appears nowhere else and
there are $k-1$ colors in $N(v)$

$\rightarrow$ consider $(k-1)$-coloring on $G-v$
If we add back v and not
all $k-1$ colors show in $N(v)$,
we would have a $(k-1)$-coloring
on the original G
$\hookrightarrow$ assign "lowest" color not in $N(v)$
to get $(k-1)$-coloring on G

contradiction

Similarly: $\forall e = (u,v) \in E(G)$
$\rightarrow$ Every proper $(k-1)$-coloring
on $G-e$ gives $c(u) = c(v)$

→ If not, we would trivially have a $(k-1)$-coloring on G

Connectivity of k-color-critical graph G

Show: G is $(k-1)$-edge-connected

To do so, first show:

For G' s.t. $\chi(G') > k$, let

$\{X, Y\}$ be a partition of $V(G')$

If $G'[X]$ and $G'[Y]$ are k-colorable

then $|[X, Y]| \geq k$

induced subgraph of G' on $X \subseteq V(G')$

size of cut

consider $X_1 X_2 \ldots X_k$

and $Y_1 Y_2 \ldots Y_k$

as independent sets defined

as independent sets defined by our k-colorings

Show: if $|[X,Y]| < k$, $\exists X_i Y_j$ that we can combine to form a k-coloring on original G'

Assume $|[X,Y]| < k$

Construct H as a bigraph

$V(H) = \{$each X_i and Y_i coarsened to a single vert$\}$

$E(H) = \{(X_i, Y_j)$ for all i,j pairs where no edge exists between some $x \in X_i$ and $y \in Y_j$ on the original $G'\}$

Note: H has more than $k(k-1)$ edges

$\rightarrow k^2$ possible, but cut $< k$

Note 2: m vertices cover at most $m * k$ edges in H

$\hookrightarrow$ $E(H)$ is not covered by only $(k-1)$ vertices

$\Rightarrow$ min cover $\geq k$

max match $\leq k$

as max match = min cover = k

If we combine all matched sets into a single color

$\Rightarrow$ we get a k-coloring on G'

contradiction

$\Rightarrow |[X, Y]| \geq k$

Bring it on home 

$\rightarrow$ show every k-critical graph is $(k-1)$-edge is $(k-1)$-edge-connected

Consider k-color-critical G

$[X,Y]$ min cut

$\to G[X]$ and $G[Y]$ are $(k-1)$-colorable

$\Rightarrow$ edge cut must be at least $(k-1)$ in size on G $\square$

More properties of G

recall:
<u>S-lobe</u>: given $S \subseteq V(G)$ and $G-S$, an S-lobe is an induced subgraph on $S+$(same component of $G-S$)

If G is k-critical, G has no min. vertex cut $\{x,y\}$ where $(x,y) \in E(G)$ AND if G has vertex cut $S=\{x,y\}$ then G has an S-lobe H where an S-lobe H where

$$\chi(H + (x,y)) = k$$

Assume we have cut $S = \{x, y\}$

and $G - S \rightarrow H_1 H_2 \ldots H_t$ as S-lobes

$\rightarrow$ each H_i are $(k-1)$-colorable

Consider $\{x, y\}$ and $c(x), c(y)$ on H_i

$\rightarrow$ on some H_i : $c(x) \neq c(y)$ otherwise we can create a $(k-1)$-coloring on G

$\hookrightarrow$ we can combine all $(k-1)$-colored S-lobes, as we have no dependencies between them except for x, y

Note: this same logic implies that at least one H_i has $c(x) = c(y)$

$\rightarrow$ so x, y are not adjacent

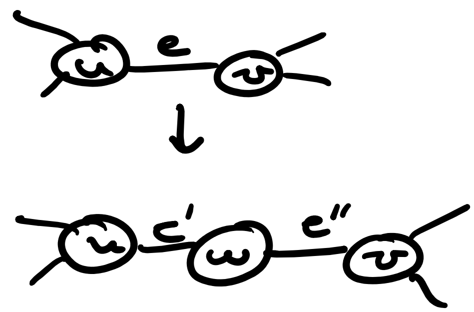
Consider H_i where $c(x) = c(y)$

$\rightarrow$ adding edge (x, y), we require a new color for x or y

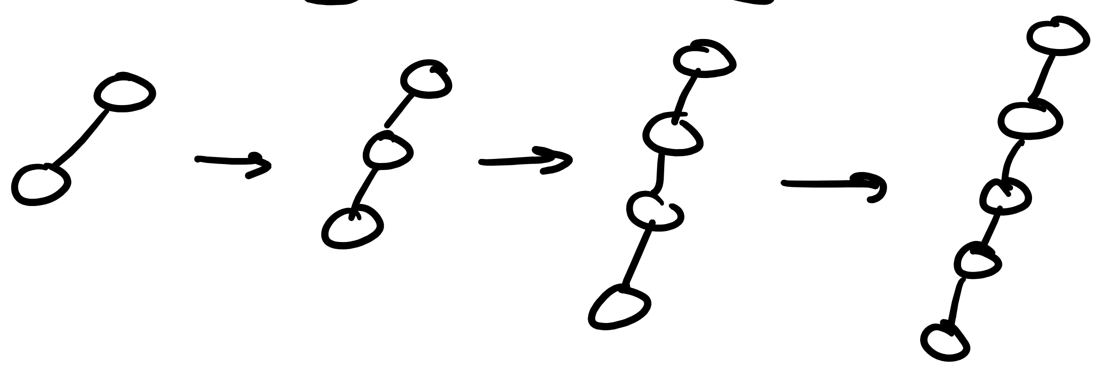
$\Rightarrow \chi(H + (x,y)) = k$ □

Subdivisions and coloring

Recall: subdivision

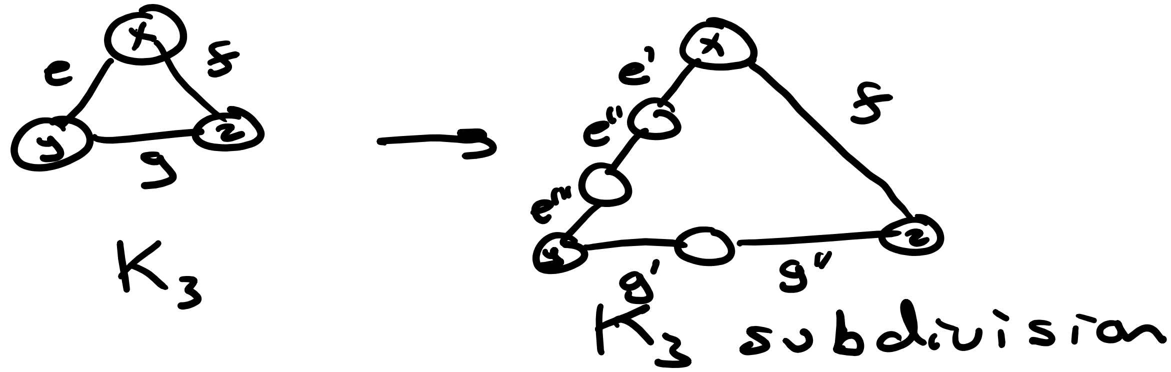


Note: we can subdivide an edge any arbitrary number of times



Note 2: we can also subdivide edges of some (sub)graph G to create a "G subdivision"

To create a G subdivision



→ we replace edges with paths

Prove: Every graph G where $\chi(G) \geq 4$ has a K_4-subdivision

We'll do strong induction on $|V(G)|$

Basis $P(4)$: K_4 and trivially

$P(n > 4)$ we have some G s.t. $\chi(G) \geq 4$

→ consider a 4-critical subgraph of G, $H \subseteq G$, $\chi(H) = 4$

Case 1: H is 2-connected

↳ assume we have $S = \{x, y\}$ where $(x, y) \in E(G)$

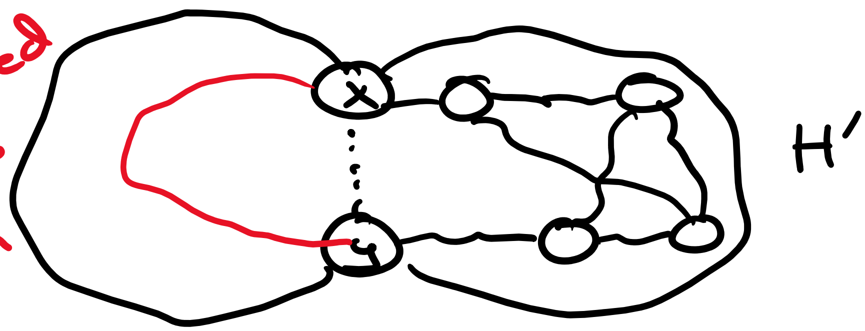
Consider S-lobe H' of H

Consider S-lobe H' of H

where $\chi(H' + (x,y)) \geq 4$

We take $H' + (x,y)$ as $P(k)$ and get a K_4 subdivision via I.H.

we need to show $\exists$ x,y-path



→ trivial → as the S-lobes of G are connected and we have at least 2

Case 2: H is at least 3-connected

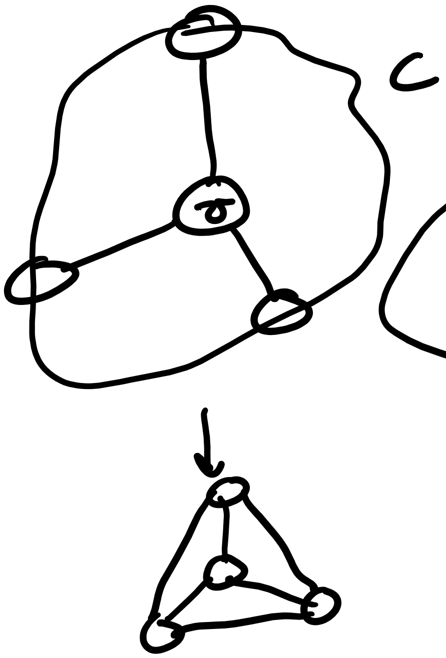
Consider $H - v$, $v \in V(H)$

→ as $H - v$ is at least 2-connected, we have a cycle of at least length 3

Recall: v, U-fan

→ a k-connected G

Recall: v, U fan



→ a k-connected G has a v, U-fan where where $|U| \geq k$

→ we consider v, C fan which has at least 3 idps from v

→ we can use those paths to construct a K_4-sub. along with C □

$\Rightarrow$ Any G with $\chi(G) \geq 4$ will contain a K_4-subdivision □