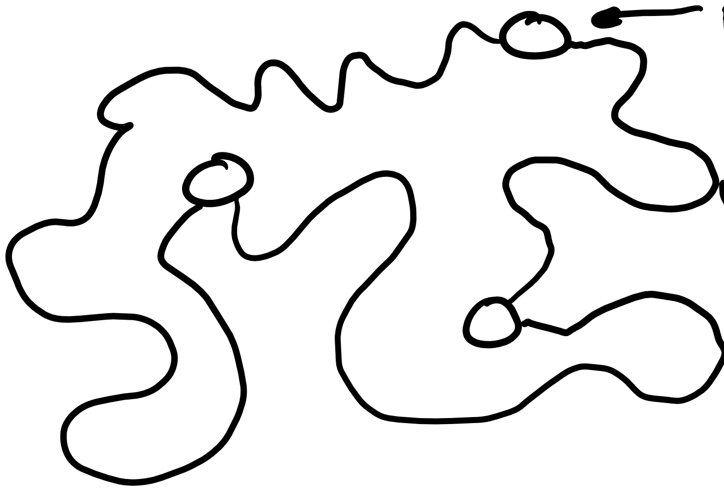
Planarity and Drawing

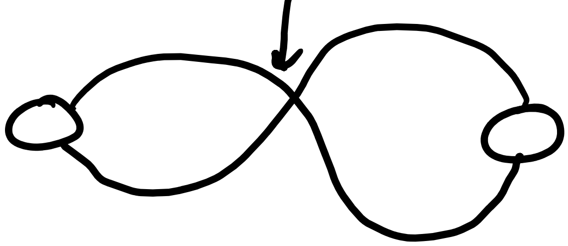
Graph drawing: a mapping of vertices and edges to points on the plane and curves between those points



points, i.e. vertices

curves, i.e. edges

Graph Planarity: a graph is planar if it can be drawn without any edge crossings



an intersection of (at least) two edge curves

down

curves

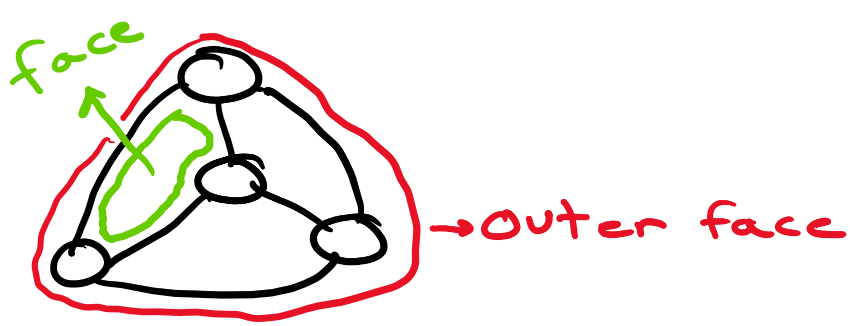
slap down

Couple more definitions:

planar embedding: a specific drawing of some graph w/no edge crossings

face: a maximal area in some embedding fully enclosed by edge curves

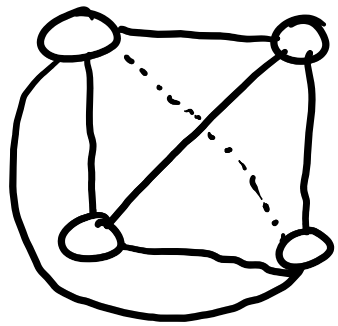
outer face: the external or unbounded face of some embedding



Note: K_4 is planar

Q: Can we draw K_4 s.t. all

Q: Can we draw K_4 s.t. all vertices are on the outer face?

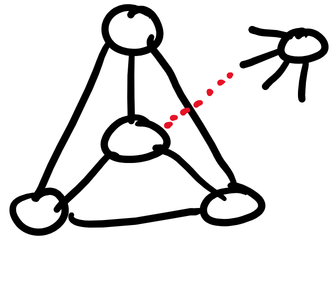


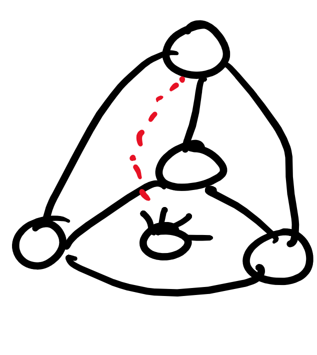
→ <u>No</u>

<u>outerplanar</u>: a graph with some embedding where all vertices are on the outer face

<u>Think about</u>: what does K_4 being not outerplanar mean for K_5?

→ we need one more vertex and 4 edge curves to draw K_5





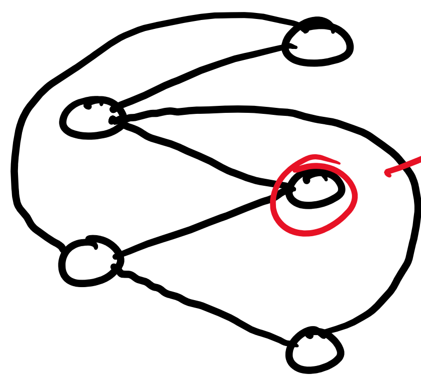
$\Rightarrow K_5$ is not planar

aka nonplanar

no embedding without crossings exists

think about: (part 2)

What about $K_{2,3}$ and $K_{3,3}$?



$K_{2,3}$ is not outerplanar

$\Rightarrow K_{3,3}$ is also nonplanar

Special subgraphs: K_4, K_5, $K_{2,3}$, $K_{3,3}$

(more later)

Other examples:

C_n: planar and outerplanar

K_3: also planar and outerplanar

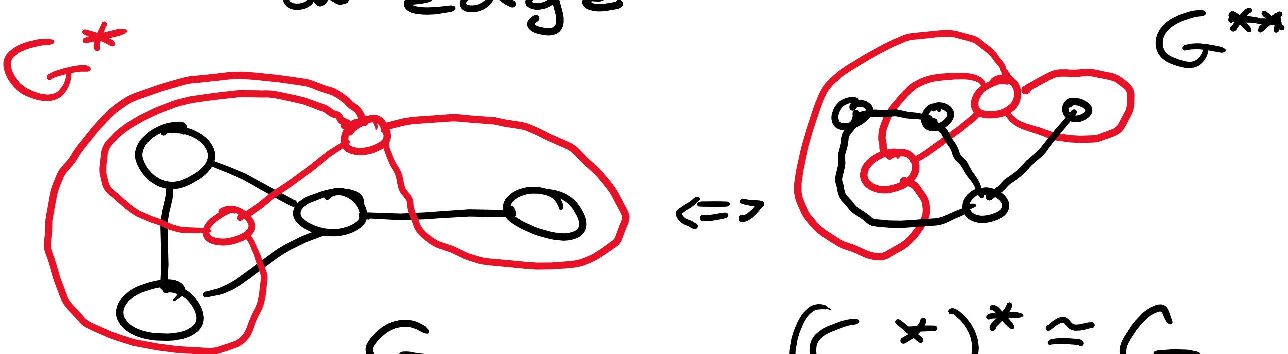
P_n: same

$K_{n\geq 5}$: nonplanar

Trees: planar & outerplanar

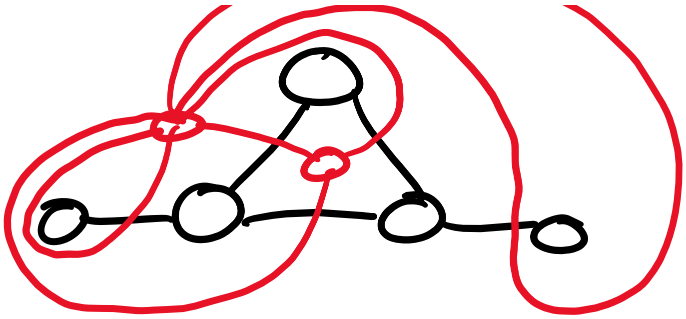
Dual Graphs

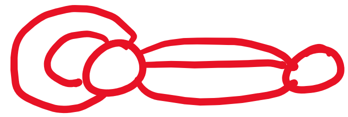
<u>Dual graph</u> G^* of some embedding of planar G is a graph whose vertices are the faces of G and edges are defined between the faces of G that share an edge



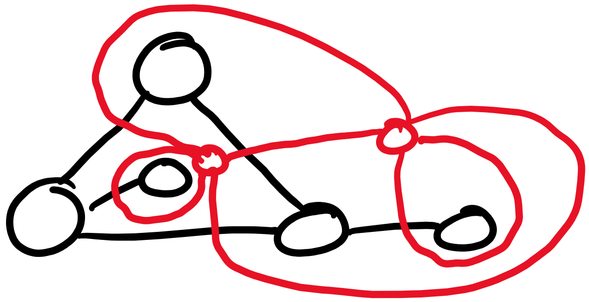
$(G^*)^* \cong G$

Note: G^* depends on a specific embedding of G





G_1^*

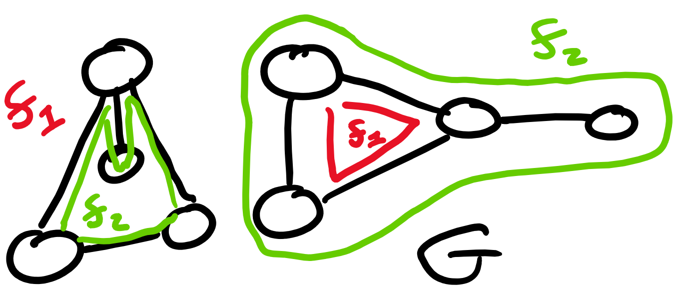


G_2^*

$G_1^* \not\cong G_2^*$

specific embeddings of G^*

Let's Talk More on faces



G has 2 faces

The <u>length</u> of a face is equal to the number edges on a minimal walk fully bounding the face

$\ell(f_1) = 3$

length $\ell(f_2) = 5$

Note: each edge contributes

Note: each edge contributes +2 to the sum of the lengths of faces for some G embedding

$$\sum_i \ell(f_i) = 2|E(G)|$$

Proofy Business  ← tie

planar G is bipartite $\Leftrightarrow$ all faces in an embedding are even in length

$\Leftrightarrow$ G^* is Eulerian

G is bipartite $\Rightarrow$ all faces $\Rightarrow$ even

Note: all possible closed walks are even

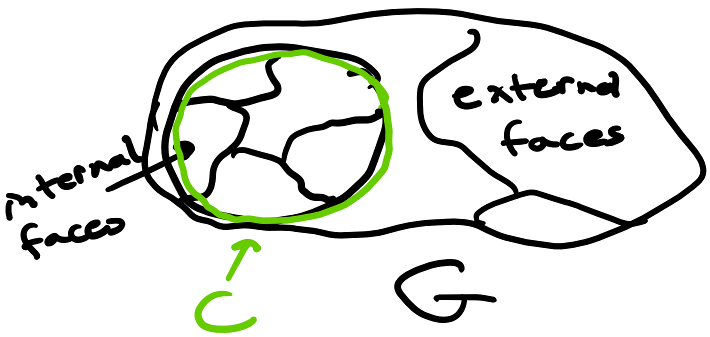
→ faces defined by closed walk lengths

$\Rightarrow$ all faces even

faces = even => G is bipartite

Consider some cycle in a
(G trivially bipartite w/ no cycle)
embedding of G

→ all of G is either internal or external to that cycle



Consider the internal portion of G
→ all faces even
$\sum_{i \in G_{int}} \ell(f_i) = \text{even}$

Note: each internal edge is counted twice

Note: each edge on C is counted once

parity

=> the cycle C must be even

=> this applies to any $C \subseteq G$

→ this applies to any $C \subseteq G$

$\Rightarrow$ G is bipartite $\square$

Proovin'

all faces even $\Leftrightarrow$ G^* is Eulerian

($\Rightarrow$) Note that the degrees of vertices in G^* is a function of the length of the face they represent in G's embedding

→ all vertices in G^* are even degree

→ and G^* will always be connected

$\Rightarrow$ so G^* is Eulerian

($\Leftarrow$) same basic logic, as G^* in Eulerian graph, this implies even degrees in G^* which implies even face lengths in G's embedding $\square$

Euler's formula

(for a connected planar embedding)

$$n - e + f = 2$$

$n \uparrow$ $|V(G)|$ $\qquad$ $e \uparrow$ $|E(G)|$ $\qquad$ $f \uparrow$ # of faces in G's embedding

we'll prove this using the

Power

of induction on $|V(G)|$

Basis: $P(1) \rightarrow$  K_1

$n=1$
$e=0$
$f=1$

$1-0+1=2$ ✓

$n=1$
$e=e$
$f=e+1$

$1-e+e+1=2$ ✓

Consider our $P(n)$ case

$\rightarrow$ there exists some edge that

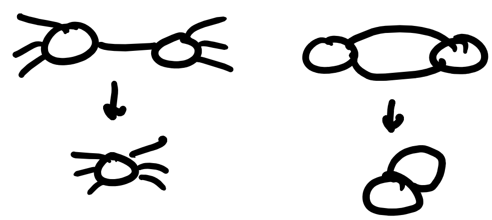
→ there exists some edge that isn't a self loop

→ contract that edge e

$$P(k) = P(n) \cdot e$$

Note: edge contraction won't make a planar graph nonplanar

I.H. on $P(k)$



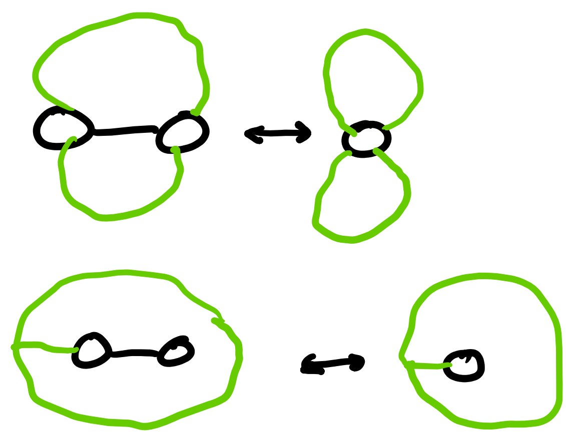
↳ $n' - e' + f' = 2$

Bring it on back to $P(n)$

$$n = n' + 1$$

$$e = e' + 1$$

$$f = f'$$



$$n' - e' + f' = 2$$

$$(n-1) - (e-1) + f = 2$$

$$n - \cancel{1} - e + \cancel{1} + f = 2$$

$$n - e + f = 2 \quad \square$$

QED

QED

Let's put EF to use

(Euler's Formula)

If G is a simple connected planar graph with $|V(G)| \geq 3$

$\Rightarrow e \leq 3n - 6$

G is simple $\rightarrow \ell(f_i) \geq 3$

From our face sum formula

$$2e = \sum \ell(f_i) \geq 3f$$

$$2e \geq 3f$$

Consider $n - e + f = 2$

$$(e = n + f - 2) \cdot 3$$

$$3e = 3n + 3f - 6$$

$$3e \leq 3n + 2e - 6$$

$-2e \qquad\qquad -2e$

$$\boxed{e \leq 3n - 6}$$

This is an upper bound on $|E(G)|$ with respect to $|V(G)|$ for a simple graph to be planar

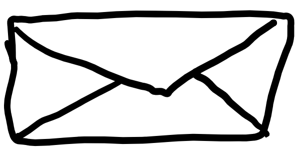
What if G is triangle free?

$$\ell(f_i) \geq 4$$

plug n' chug

...

$$\boxed{e \leq 2n - 4}$$

Note: 

These conditions are necessary but NOT sufficient

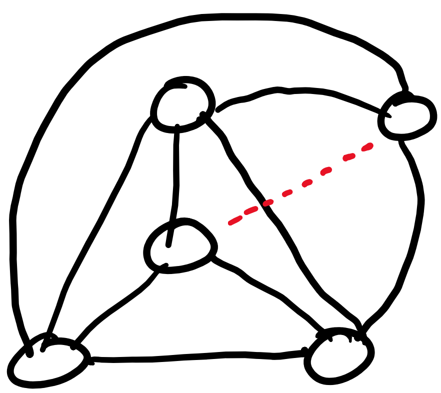
all planar graphs have these conditions

if a G has these conditions, then it is planar $\Big\}$ → not the case here

Let's get EXXXXTREEEEMEE

maximal planar G: adding any edge to planar G will make it nonplanar

minimal nonplanar G: deleting any edge from nonplanar G will make it planar



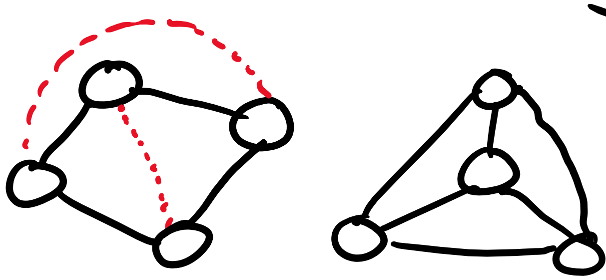
→ K_5 is minimal nonplanar

triangulation: a planar embedding where all faces are length 3

$$\boxed{\text{maximal planar} \iff \text{triangulation}}$$

↑ note bounds above

Also note: an edge can trivially be drawn in any face of length at least 4 without crossing



K_4 is a triangulation

=> maximally planar

(technically)

Next: Kuratowski's theorem