SLPT: No measurements or data exists in a vacuum.

Thus far...

We've identified some **necessary** conditions for graph planarity

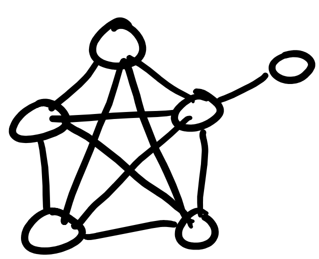
$e \le 3n-6$

$e \le 2n-4$ if $G$ is triangle-free

$G$ has no $K_5$ subdivision

$G$ has no $K_{3,3}$ subdivision

**Not** sufficient



$e = 11$

$n = 6$

$11 \le 3 \cdot 6 - 6 = 12$ ✓

Are these conditions sufficient? (taken together)

Kuratowski: Yes

$K_5$ & $K_{3,3}$ → Kuratowski Subgraphs (K.S.)

Note: If subgraph $H \subseteq G$ is nonplanar, then any subdivision of $H$ is nonplanar

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Kuratowski's Theorem

$G$ is planar iff $G$ has no Kuratowski subgraphs

($\Rightarrow$) pretty trivial

($\Leftarrow$) Buckle up

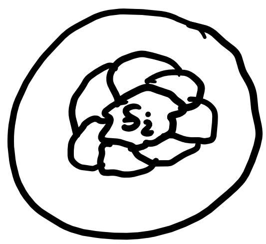
(does there exist a counter-example?)

① For every face $f_i$ of a planar embedding of $G$, $\exists$ an embedding with $f_i$ as the outer face

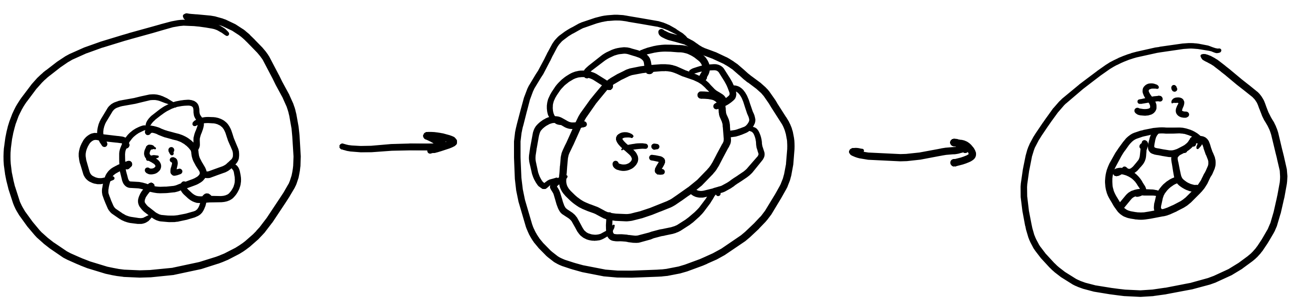
(AKA: any $v \in V(G)$ or $e \in V(G)$ can

(AKA: any $v \in V(G)$ or $e \in V(G)$ can be drawn on the outer face of $G$)

Consider embedding $G$ on a sufficiently large sphere




→ expand $f_i$ and return a projection of $G$ bounded by $f_i$




$\Rightarrow f_i$ is the outer face ✓

② Every minimal nonplanar graph $G$ is 2-connected

→ $\forall H \subset G$, $H$ is planar

Assume $\exists v \in V(G)$: $G - v$ is disconnected

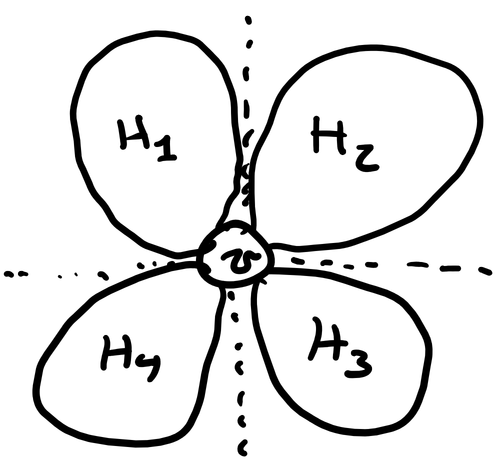
Assume $\exists v \in V(G): G - v$ is disconnected

(Assume $G$ is only 1-connected)

$$G - v = \underbrace{H_1 H_2 \ldots H_k}_{\text{Components of } G-v} \quad \text{(all } H_i \text{ are planar)}$$

We can create an embedding of $G$ by "squeezing" all of $H_i(+v)$ into $\frac{360°}{k}$ around $v$

From ① for all $H_i + v$, there exists an embedding with $v$ on the outer face

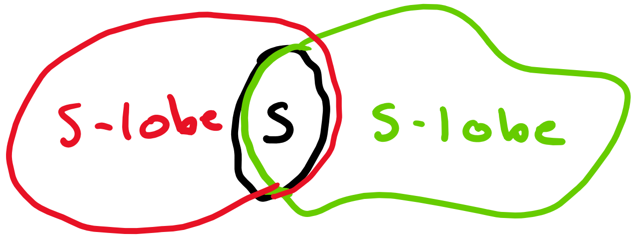



Contradiction on a 1-connected $G$

$\Rightarrow$ $G$ must be 2-connected ✓

③ Recall: S-lobes are an induced

Recall: S-lobes are an induced subgraph on some vertex cut S and some component of G−S




Let $S = \{x, y\}$ be a vertex cut on some 2-connected G

If G is nonplanar ⟹ adding edge $(x,y)$ to some S-lobe of G yields a nonplanar graph

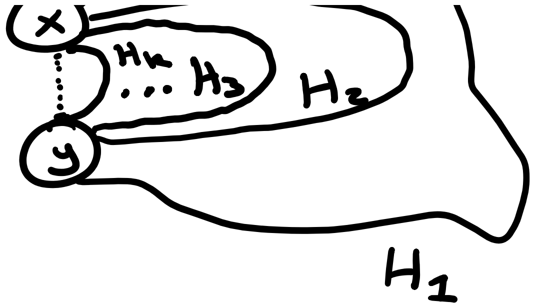
define: $H_i = G_i \cup \{x,y\} \cup (x,y)$

S-lobe, comp. of G−S, S, added edge

From ①, $H_i$ has an embedding where $(x,y)$ is on the outer face

(Assuming $H_i$ is planar)


→ We can iteratively embed all $H_{i=2 \dots k}$ within




embed all $H_{i=1\ldots k}$ within the internal face of $H_{i-1}$ containing edge $(x,y)$

Contradiction

→ we have at least one $H_i$ as nonplanar ✓

④ If G is a graph with the fewest edges among all nonplanar graphs w/o a K.S. ⇒ G must be 3-connected

Note: G doesn't exist, but if it did then it must be 3-connected

Why: restrict any possible counter-example of Kuratowski to 3-connected graphs

Note 2: deleting an edge cannot create a K.S.

$\rightarrow \forall e \in E(G)$ : $G - e$ is planar and does not have K.S.

From (2) $\rightarrow$ G is 2-connected

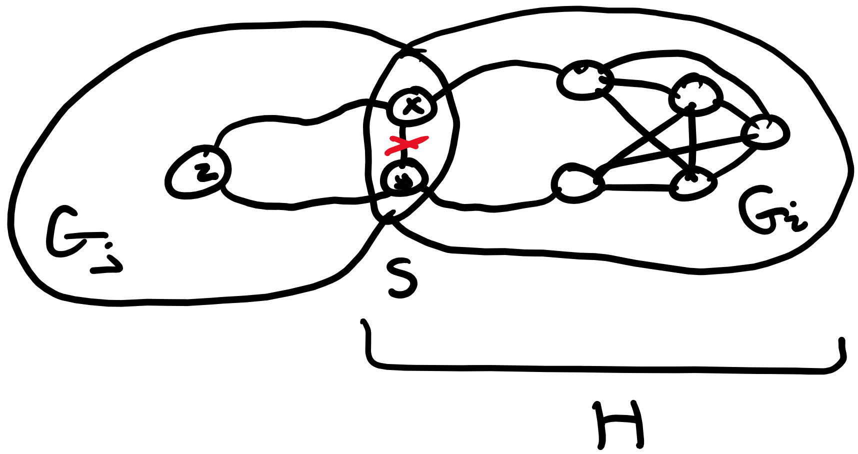
Assume $\exists S = \{x, y\}$ then some S-lobe of $G_i + S$ is nonplanar (From (3))

$\rightarrow$ define H as that S-lobe $+ (x, y)$

From our minimality condition:

as $|E(H)| < |E(G)| \rightarrow$ H must have a K.S.

Consider the above configuration




However, since G is 2-connected,

$\exists z \in V(G_j) : G_i \neq G_j$, where we

$\exists z \in V(G_j) : G_i \neq G_j$, where we have 2-disjoint $z,x$ and $z,y$-paths (from our $U,v$-fan theorem)

**Contradiction**

$\Rightarrow$ we would still have a K.S. in $G$

$\Rightarrow$ Any minimum counter-example must be 3-connected ✓

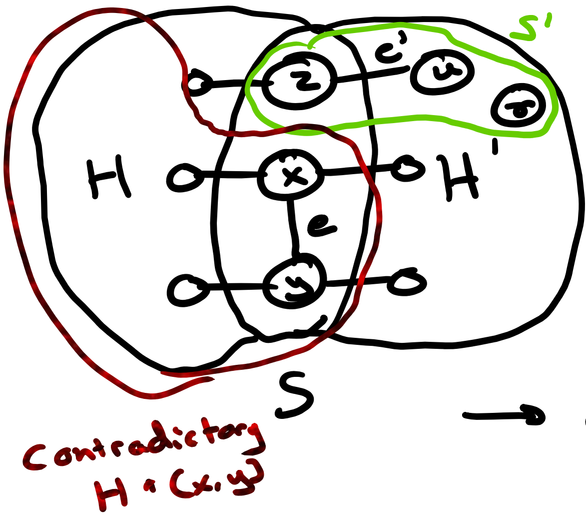
Next up: show all 3-connected graphs w/o a K.S. $\Rightarrow$ planar

---

⑤ If $G$ is 3-connected and $|V(G)| \geq 5$ $\exists e \in E(G)$ s.t. $G \cdot e$ is also going to be 3-connected

Consider $e = (x,y) \in E(G)$ s.t. $G \cdot e$ is not 3-connected

$G \circ e$ is not 3-connected




First: assume there does not exist any $e$ s.t. $G \circ e$ is 3-connected

→ all edges are are within a cut of size 3 with same 'mate' vertex $z$ (as drawn)

Select $S = \{x, y, z\}$ s.t. $|V(H)|$ is maximum

Each of $x, y, z$ have neighbors in each of $H$ and $H'$

- consider $u \in \{N(z) \cap V(H')\}$
- consider $v$, the mate of $(u, z)$

$G - \{x, y, z\}$ is disconnected

$V(H) \cup \{x, y\}$ is connected and within a component of $G - \{z, u, v\}$

Contradiction

Contradiction

↳ on our selection of H

$\Rightarrow \exists e \in E(G)$ s.t. $G \cdot e$ is 3-connected ✓

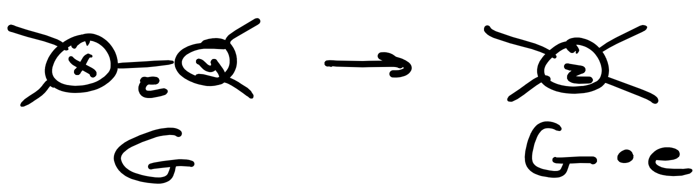
⑥ If G has no K.S.,

$\Rightarrow G \cdot e$ has no K.S.

contrapositive

$G \cdot e$ has a K.S. $\Rightarrow$ G has a K.S.

- define H as K.S. in $G \cdot e$
- define $z \in V(G \cdot e)$, $z \leftarrow e = (x,y)$, $e \in E(G)$

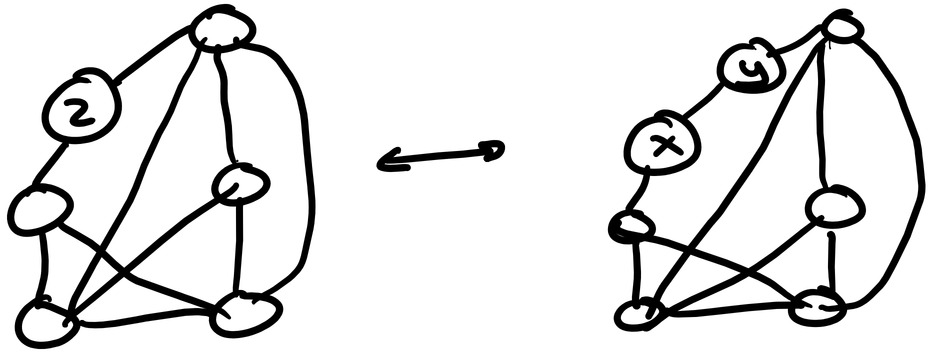



- Assume $z \in$ K.S., consider the possible degrees of $z, x, y$ w.r.t. H

Case 1: $d_H(z) < 3$

(z's degree in KS. H)

→ z is along a subdivided edge




Case 2: $d_H(z) \geq 3$

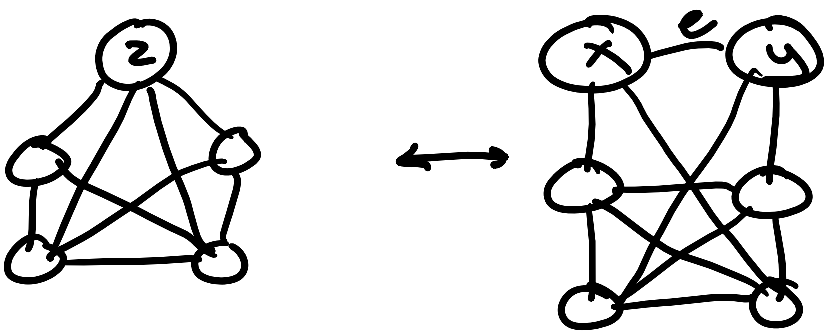
$d_H(x) \leq 2$ and $d(y) \leq 2$

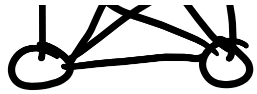
→ same thing, e is along a subdivided edge

Case 3: $d_H(z) \geq 3$

$d_H(x) \geq 3$ and $d_H(y) \geq 3$

$K_5 \leftrightarrow K_{3,3}$ is the only way





$\Rightarrow$ for all cases we observe that edge contraction/expansion retains a K.S.

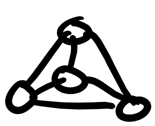
AND

$\Rightarrow$ Edge contraction does not create a K.S. ✓

# Bring it on Home via Induction

(7) If $G$ is 3-connected with no K.S. $\Rightarrow G$ has an embedding on the plane

Induction on $|V(G)|$

Basis: $K_4$  is planar ✓

Consider our $P(n)$ case from (5)

Consider our P(n) case

from (5)

$\rightarrow \exists e$ s.t. $P(n) \cdot e$ is 3-connected

Note: $G \cdot e$ has no K.S. from (6)

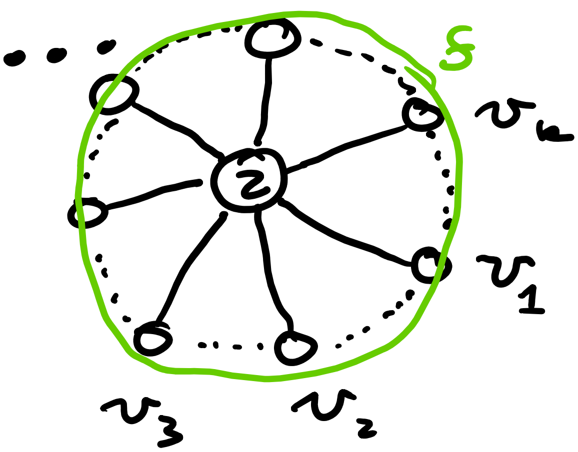
$P(k) = P(n) \cdot e$

I.H. on $P(k)$ gives us an embedding on the plane

Bring it back to $P(n)$

Consider $z \leftarrow (x, y) = e$ ((contraction))

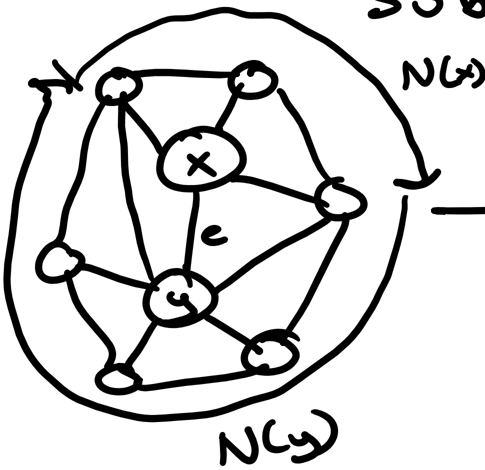
Note: all $N(z)$ can form a face containing vertex $z$




- order all $N(z)$ as $v_1 v_2 \dots v_k$, $k = d(z)$
- consider an embedding $z \rightarrow (x, y)$ (uncontract)

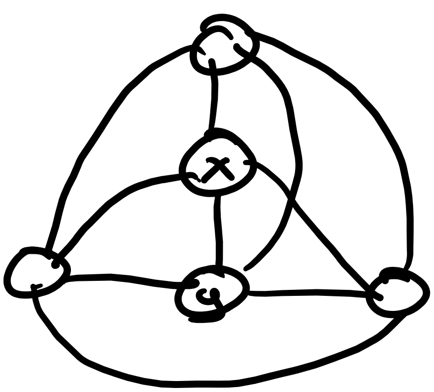
Case 1: $N(x)$ is some exclusive subset of ...

Case 1: $N(x)$ is some exclusive subset of $v_1 \dots v_k$




$\rightarrow$ trivial to construct an embedding

Case 2: $|N(x) \cap N(y)| \geq 3$

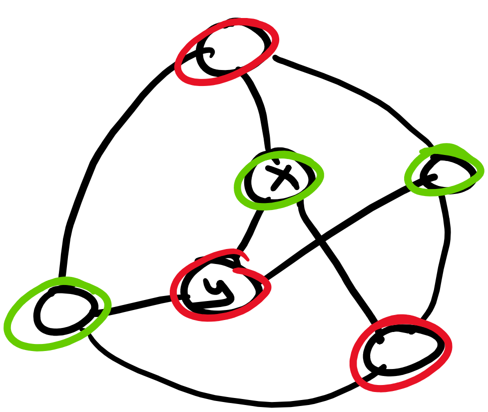



$\rightarrow$ we have $K_5$ K.S.

Case 3: $N(x)$ alternates with $N(y)$ s.t. $v_i v_j \in N(x)$

$v_l v_m \in N(y)$

$v_i < v_l < v_j < v_m$




$\rightarrow K_{3,3}$ K.S.

4 + 7 = Kuratowski's Theorem

$\downarrow$ any minimum counter-example is 3-connected

$\searrow$ any 3-connected counter-example doesn't exts

QED ☑