

## 19.1 Conditions for Planarity

So far, we've come up with a few conditions to determine whether or not a graph  $G$  is planar. We've observed that  $K_5$  does not have a planar embedding. Similarly, neither does  $K_{3,3}$ . So obviously, any graph that has  $K_5$  or  $K_{3,3}$  as a subgraph is not planar. Additionally, we've used Euler's formula to show how all planar graphs have  $m \leq 3n - 6$ , where  $m = |E(G)|$  and  $n = |V(G)|$ . When  $G$  is triangle-free, then  $m \leq 2n - 4$ .

So for now, we know that a graph is not planar if:

1. It has  $K_5$  as a subgraph
2. It has  $K_{3,3}$  as a subgraph
3.  $m > 3n - 6$
4.  $m > 2n - 4$  if  $G$  is triangle-free

Note that in terms of determining if  $G$  is planar, we've only shown that these conditions are necessary but not sufficient. E.g., a graph with  $m \leq 3n - 6$  is not necessarily planar – think of  $K_5 + v$ , where  $v$  is a single additional vertex attached by a single edge to some  $u \in K_5$  ( $m = 11$ ,  $n = 6 \rightarrow 11 < 12$ , but we know a graph with a  $K_5$  subgraph can't be planar).

Let's explore further conditions. Recall a subdivision, which is created by replacing a single edge with a path. Note that subdividing an edge does not affect planarity, since an embedding of a subdivided edge can be used to create an embedding of the original graph and vice-versa. Therefore, we can see that a planar graph cannot contain a subgraph that is a subdivision of  $K_5$  or  $K_{3,3}$ . These subgraphs, subdivisions of  $K_5$  and  $K_{3,3}$ , are called **Kuratowski subgraphs**.

## 19.2 Kuratowski's Theorem

**Kuratowski's Theorem** is the much stronger statement that a graph is planar if and only if it does not contain a subdivision of  $K_5$  or  $K_{3,3}$ . To prove Kuratowski's Theorem, we'll show the following:

1. For every face  $F_i$  of an embedding of  $G$ , it's possible to draw a new embedding of  $G$  with  $F_i$  as the outer face.
2. A **minimal nonplanar graph** is a nonplanar graph such that any subgraph is planar. Every minimal nonplanar graph is 2-connected.

3. An  **$S$ -lobe** of  $G$  is an induced subgraph consisting of a vertex set  $S$  as well as the vertices of some component of  $G - S$ . If  $S = \{x, y\}$  is a separating set of nonplanar graph  $G$ , then adding the edge  $e = (x, y)$  to some  $S$ -lobe of  $G$  yields a nonplanar graph.
4. If nonplanar graph  $G$  has the fewest edges among all nonplanar graphs without Kuratowski subgraphs, then  $G$  is 3-connected.
5. Every 3-connected graph  $G$  with at least five vertices has an edge  $e$  such that  $G \cdot e$  is 3-connected.
6. If  $G \cdot e$  has a Kuratowski subgraph, then  $G$  has a Kuratowski subgraph.
7. A convex embedding of a graph is an embedding where each face is a convex polygon. If  $G$  is 3-connected with no subdivision of  $K_5$  or  $K_{3,3}$ , then  $G$  has a convex embedding on the plane with no 3 vertices in a line.

If  $G$  has a convex embedding, then obviously it must be planar. Therefore, any graph that contains a Kuratowski subgraph is nonplanar.